Tutorial 11.

The dual space $V' := \underline{\underline{\mathcal{L}(V;\mathbb{K})}} = b_1\alpha_1 + b_2\alpha_2$

$V=\mathbb{R}^2$ basis e_1, e_2.

there is a "dual basis" to $\{e_1,e_2\}$ we call $\{\alpha_1,\alpha_2\}$

$$\alpha_1(e_1)=1 \qquad \alpha_1(e_2)=0 \qquad \alpha_1(a_1e_1+a_2e_2) = a_1\alpha(e_1)+a_2\alpha(e_2)$$
$$= a_1$$

$$\alpha_2(e_1)=0 \qquad \alpha_2(e_2)=1 \qquad \alpha_2(a_1e_1+a_2e_2)=a_2$$

$\{e_k\}$ of V $\rightsquigarrow$ $\{\alpha_k\}$ of V' defined by $\alpha_i(e_j)=\delta_{ij}$

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$A\in V'$ $\qquad A(a_1e_1+a_2e_2) = A(a_1e_1)+A(a_2e_2)$

$$= a_1\underline{A(e_1)} + a_2\underline{A(e_2)}$$

$\beta = A(e_1)\alpha_1 + A(e_2)\alpha_2$

$$\beta(a_1e_1+a_2e_2) = A(e_1)\alpha_1(a_1e_1+a_2e_2) + A(e_2)\alpha_2(a_1e_1+a_2e_2)$$
$$= a_1A(e_1) + a_2A(e_2)$$

$A=\beta$.

$A\begin{pmatrix}a_1\\a_2\end{pmatrix} = 2a_1 - a_2$ $\qquad$ so $A\in\mathcal{L}(\mathbb{R}^2;\mathbb{R})$

$A = 2\alpha_1 - \alpha_2$

For finite dimensional vector spaces, $(V')'=V$

If $v\in V$ is actually $\in(V')'$ how does v act on $\alpha\in V'$?

$v(\alpha) = \alpha(v)$

$v(a\alpha+b\beta) = (a\alpha+b\beta)(v) = a\,\alpha(v) + b\,\beta(v) = a\,v(\alpha) + b\,v(\beta)$

"Bilinear map" = 2-linear maps.

$B(a v, bw) = ab\, B(v,w)$

$B(v+v', w) = B(v,w) + B(v',w)$.

$B(v, w+w') = B(v,w) + B(v,w')$.

$V_1 = \mathbb{R}^2 \quad V_2 = \mathbb{R}^3 \quad W = \mathbb{R}. \qquad A \in \mathcal{L}(V_1, V_2; W)$

$\{e_1, e_2\} \quad \{f_1, f_2, f_3\} \quad 1$

$A(a_1 e_1 + a_2 e_2, \ b_1 f_1 + b_2 f_2 + b_3 f_3)$

$= a_1 A(e_1, b_1 f_1 + b_2 f_2 + b_3 f_3) + a_2 A(e_2, b_1 f_1 + b_2 f_2 + b_3 f_3)$

$= a_1 b_1 \underline{A(e_1, f_1)} + a_1 b_2 \underline{A(e_1, f_2)} + a_1 b_3 \underline{A(e_1, f_3)}$

$+ a_2 b_1 \underline{A(e_2, f_1)} + a_2 b_2 \underline{A(e_2, f_2)} + a_2 b_3 \underline{A(e_2, f_3)}$

Special multilinear maps. Let $\{\alpha_1, \alpha_2\}$ basis of V_1' dual to $\{e_k\}$
$\{\beta_1, \beta_2, \beta_3\}$ be the basis V_2' dual to $\{f_k\}$

Define $\alpha \otimes \beta \in \mathcal{L}(V_1, V_2; \mathbb{K})$ by. $\alpha \in V_1' \quad \beta \in V_2'$

$$\alpha \otimes \beta\, (v_1, v_2) = \alpha(v_1)\, \beta(v_2)$$

$$\alpha_1 \otimes \beta_2 (\overset{V_1}{2e_1 - e_2}, \overset{V_2}{3f_1 + f_2}) = \alpha_1(2e_1 - e_2)\, \beta_2(3f_1 + f_2)$$
$$= 2 \cdot 1 = 2$$

$\uparrow \times 2$

$$\alpha_1 \otimes \beta_2 (4e - 2e_2, 3f_1 + f_2) = \alpha_1(4e_1 - 2e_2)\, \beta_2(3f_1 + f_2)$$
$$= 4$$

$$\alpha_i \otimes \beta_j (e_k, f_\ell) = \alpha_i(e_k)\, \beta_j(f_\ell) = \begin{cases} 0 & \text{if } i \neq k \text{ or } j \neq \ell \\ 1 & \text{if } i = k \ \underline{\text{and}}\ j = \ell. \end{cases}$$

$\alpha_1 \otimes \beta_1 (e_1, f_1) = 1$

$(e_k, f_\ell) = 0 \qquad k \neq 1, \ell \neq 1$

$$B = A(e_1, f_1)\, \alpha_1 \otimes \beta_1 + A(e_1, f_2)\, \alpha_1 \otimes \beta_2 + \ldots + A(e_2 f_3)\, \alpha_2 \otimes \beta_3.$$

$$B(e_1,f_1) = A(e_1,f_1)\cdot 1 + 0 + 0 + 0 + 0 + 0. = A(e_1,f_1)$$
$$B(e_1,f_2) = 0 + A(e_1,f_2)\cdot 1 + 0 + 0 + 0 + 0 = A(e_1,f_2)$$
$$\vdots$$

$\Rightarrow B(v_1,v_2) = A(v_1,v_2)$ for all $v_1 \in V_1$ and $v_2 \in V_2$.

ie $B=A$

Multilinear map $\mathcal{L}(V_1,V_2;W)$ for $\dim W > 1$?.

Choose a basis for W and look at components. $w_1,\dots,w_n$

$$A(v_1,v_2) = \underline{A_1(v_1,v_2)}\,\omega_1 + \underline{A_2(v_1,v_2)}\,\omega_2$$

$$\mathcal{L}(V_1,V_2;\mathbb{K}).$$

"Normally" define tensor abstractly, show they are equivalent to multilinear map.

$$V_1'\otimes V_2' := \mathcal{L}(V_1,V_2;\mathbb{K}) \leftarrow \text{most common situation.}$$

$$V_1\otimes V_2 = (V_1')'\otimes(V_2')' = \mathcal{L}(V_1',V_2';\mathbb{K}).$$

$$\mathcal{L}(V_1,V_2,V_3;W) = \mathcal{L}\underbrace{(V_1\;;\;\mathcal{L}(V_2,V_3;W))}_{1\text{-linear}:=\text{linear}}$$

$$\dim\mathcal{L}(V_1,V_2,V_3;W) = \dim V_1 \times \dim\mathcal{L}(V_2,V_3;W)$$

$V = W = \mathbb{R}^2$

tensors. $V' \otimes W$

If we have $A \in \mathcal{L}(V, W'; \mathbb{K})$ how is this a linear $\underset{B\in}{\mathcal{L}(V,W)}$?

$A(v, \beta) = \beta(\underbrace{B(v)}_{W}) \in \mathbb{K}.$

$v_1, v_2 \quad w_1, w_2$

$\alpha_1, \alpha_2 \quad \beta_1, \beta_2$

$B(v) = \underbrace{\beta \mapsto A(v, \beta)}_{W' \to \mathbb{K}} \in W'' = W$

$B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad B(v) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} v$

$A(v, \beta) = \beta(B(v))$

$A(e_1 - 2e_2, \beta_1 + \beta_2) = (\beta_1 + \beta_2)\left(\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) = (\beta_1 + \beta_2)(0e_1 + 1e_2)$

$= 1$

Give a multilinear map in $\mathcal{L}(V, W'; \mathbb{K}) = V' \otimes W$

$A = \alpha_1 \otimes w_1 + 3\alpha_2 \otimes w_2$

$A(v_1, \beta_1) = \alpha_1(v_1) \underbrace{w_1(\beta_1)}_{W''(W')} + 3\alpha_2(v_1) w_2(\beta_1)$

$= \alpha_1(v_1)\beta_1(w_1) + 3\alpha_2(v_1)\beta_1(w_2)$

$= 1 \cdot 1 + 3 \cdot 0 \cdot 0 = 1.$

$B(v_1) = \beta \mapsto A(v_1, \beta)$

$= \beta \mapsto \alpha_1(v_1) w_1(\beta) + 3\alpha_2(v_1) w_1(\beta)$

$= \beta \mapsto 1 \cdot w_1(\beta) + 0$

$= \beta \mapsto w_1(\beta)$

$= w_1$

If all the vector spaces $V_1 = \cdots = V_n = V$ $\qquad \sigma \in S_n$

$$\sigma . \, v_1 \otimes v_2 \otimes v_3 (\alpha_1, \alpha_2, \alpha_3) = v_1 \otimes v_2 \otimes v_3 (\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \alpha_{\sigma^{-1}(3)})$$
$$= v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)} (\alpha_1, \alpha_2, \alpha_3)$$

$\alpha := \alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1 \quad \in V \otimes V.$ $\qquad S_2 = \{ \mathbb{1}, (12) \}$

$(12). \alpha = \alpha_2 \otimes \alpha_1 - \alpha_1 \otimes \alpha_2 = -\alpha$

α is antisymmetric

$\beta = \alpha_1 \otimes \alpha_1$ $\qquad$ symmetric