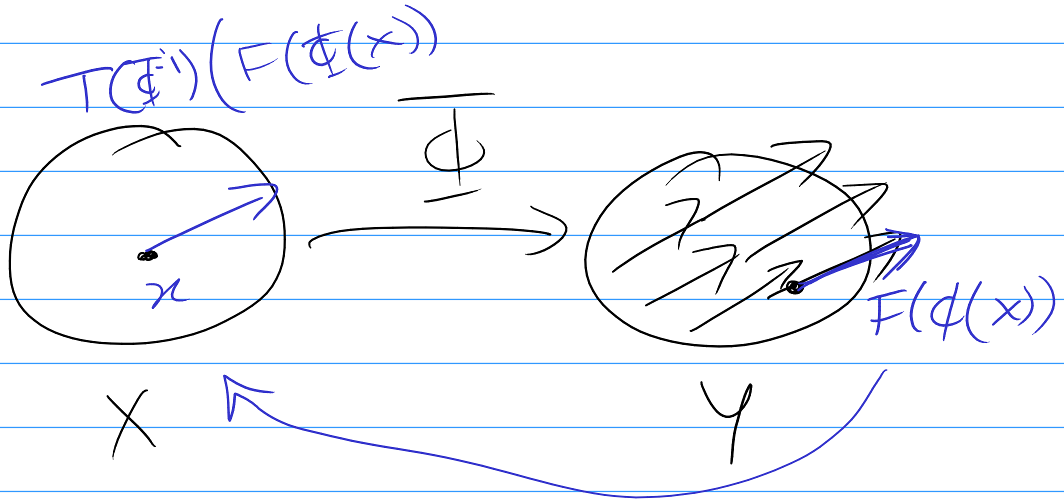
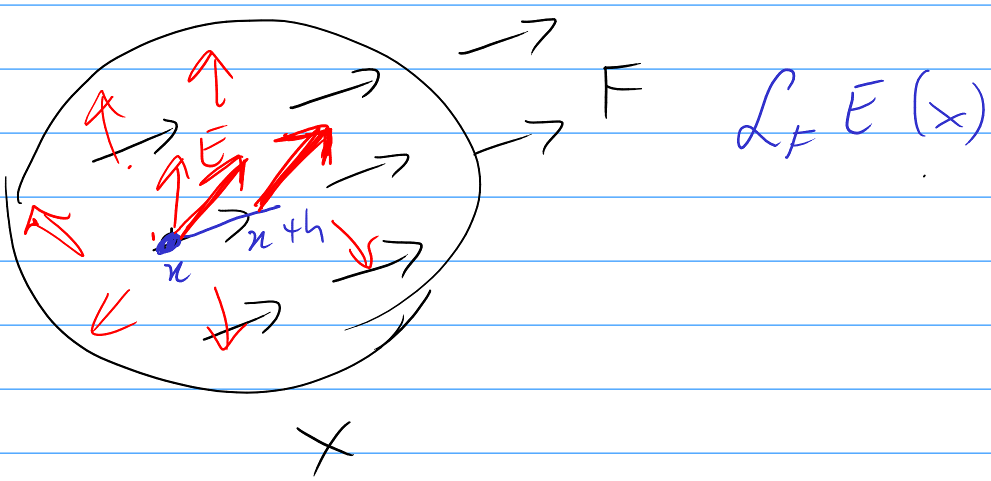


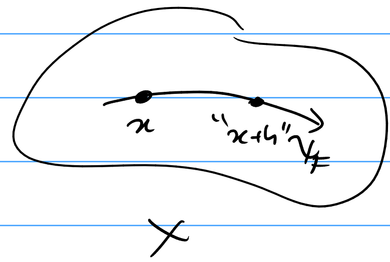

Recall, the problem with derivatives on a manifold is no vector space structure.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$




$$\mathcal{L}_F E(x) = \lim_{h \to 0} \frac{T(\psi_F(h,x)^{-1})\big(E(\psi_F(h,x))\big) - E(x)}{h}$$

Lie Derivative of a function $f : X \to \mathbb{R}$ $\quad F \in \mathrm{Vec}'(M)$

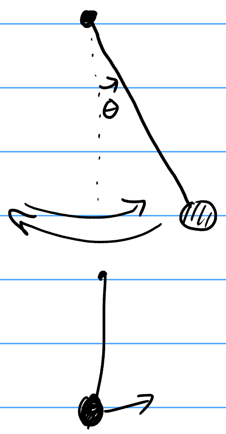



$$\mathcal{L}_F f(x) = \lim_{h \to 0} \frac{f(\psi_F(h,x)) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(\gamma(h)) - f(\gamma(0))}{h}$$

$$= \frac{d}{dt}\Big|_{t=0} f(\gamma(t)) = \Theta_F(f)$$

Example relevant to Q30.



$$\theta'' + \frac{g}{l}\sin\theta = 0. \qquad \theta \in \mathbb{R}.$$

$V = \theta' \in \mathbb{R}$ $\quad (\theta, V)$ describe the state completely.

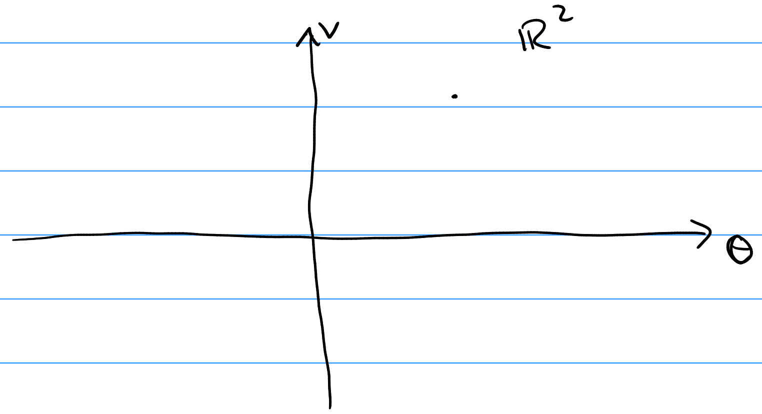



Image a path $\gamma(t) = (\theta(t), v(t))$

$$\theta' = v$$
$$v' = -\underbrace{\frac{g}{l}}_{l = 9{,}8m = g}\sin\theta$$

$$\gamma'(t) = \begin{pmatrix} v(t) \\ -\sin\theta(t) \end{pmatrix} = F(\theta, v)$$

$$F(v,\theta) = 0 \iff \begin{pmatrix} v \\ -\sin\theta \end{pmatrix} = 0 \iff v = 0,\ \theta \in \pi\mathbb{Z}$$

Energy $\quad E(\theta, v) = \frac{1}{2}M(lv)^2 + mgl(1-\cos\theta)$

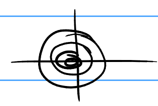
$$\tilde{E} = E/ml^2 = \frac{1}{2}v^2 + 1 - \cos\theta$$

Calculate $\theta_F(\tilde{E})$ $\quad$ hint $\quad$ write $\tilde{E}(\theta(t), v(t))$

$$= \frac{d}{dt}\Big|_{t=0} \tilde{E}(\theta(t), v(t)) = \frac{\partial \tilde{E}}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial \tilde{E}}{\partial v}\frac{\partial v}{\partial t}$$

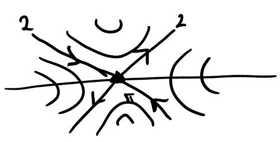
$$= \sin\theta\ \theta' + v\ v' = \sin\theta\ v + v(-\sin\theta) = 0.$$

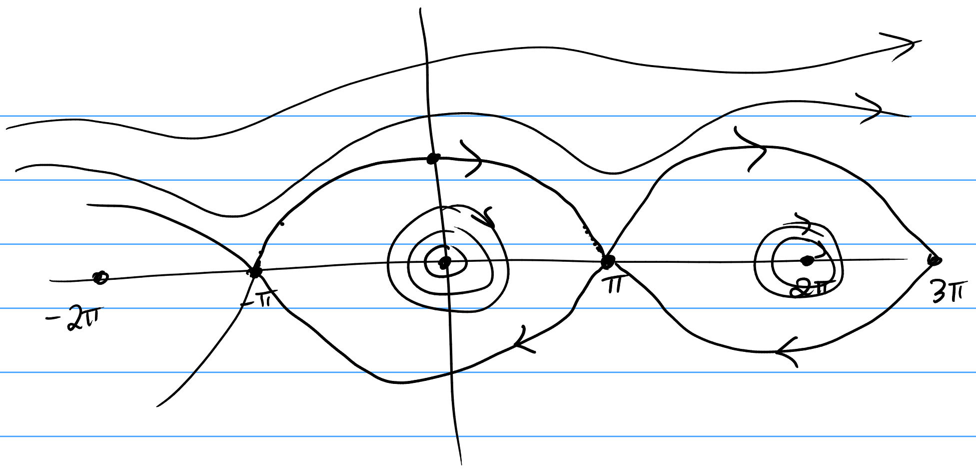
When $\theta \approx 0$, $v \approx 0$ $\quad \tilde{E} \approx \frac{1}{2}v^2 + \frac{1}{2}\theta^2$



$\theta \approx \pi$, $v \approx 0$, $\theta = \pi + \alpha$ $\quad \tilde{E} \approx \frac{1}{2}v^2 + 2 - \frac{1}{2}\alpha^2$

$$2 + \frac{1}{2}(v-\alpha)(v+\alpha)$$






3(a))

If $\gamma$ is not injective $\quad \gamma(t_1) = \gamma(t_2)$

$$= \gamma(t_1 + \underbrace{(t_2 - t_1)}_{p.})$$

Use that maximal integral curves are unique. proof. Assume $x, y$ maximal.

Define $z(t) = \begin{cases} x(t) & t \in I \\ y(t) & t \in J \end{cases}$ contradiction

is defined on $I \cup J$.

Assume $\gamma$ max int curve through $\gamma(t_1)$.

Consider $\tilde{\gamma}(t) = \gamma(t+p)$

$$\tilde{\gamma}'(t) = \gamma'(t+p) = F(\gamma(t+p))$$

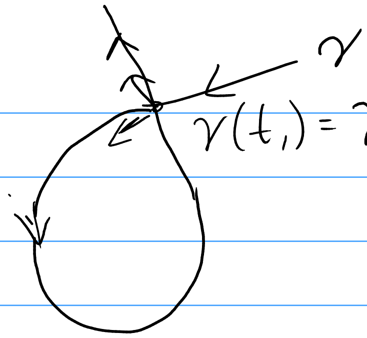
$$= F(\tilde{\gamma}(t))$$

$\tilde{\gamma}$ is an integral curve of $F$.

$\tilde{\gamma}$ is maximal.

$$\tilde{\gamma}(t_1) = \gamma(t_1 + p) = \gamma(t_1)$$

$\overset{\gamma(t+p)}{\tilde{\gamma}(t)} = \gamma(t)$ for all $t \in \mathbb{R}$.




$\gamma(t_1) = \gamma(t_2)$ not possible