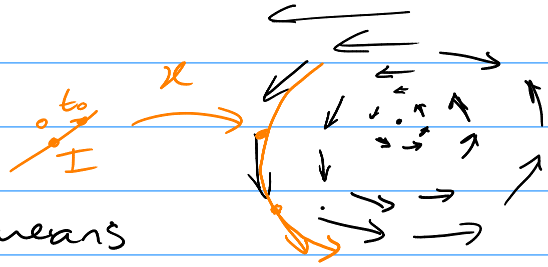
Tutorial 9

25(a). $F(x,y) = (y,-x)$ in $\mathbb{R}^2$



Integral curve $x(t)$ of F means
$\alpha(s) =$ " $s \mapsto x(t+s)$ " is $F(x(t))$

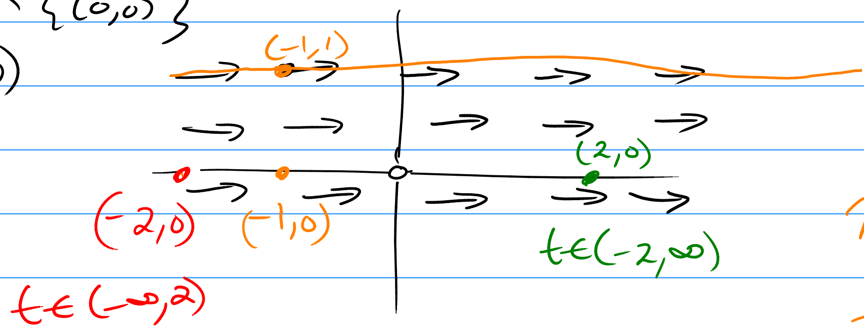
A vector at p is equivalence class α, $\alpha(0)=p$.

$$\alpha(s=0) = x(t+0) = x(t)$$

Example where integral curve exists for different amounts of time, depending on initial condition

$X = \mathbb{R}^2 \setminus \{(0,0)\}$

$F = (1,0)$



$\gamma_{(-1,1)}(t) = (-1+t, 1)$

$t \in \mathbb{R}$

$\gamma_{(-1,0)}(t) = (-1+t, 0)$

$t \in (-\infty, 1)$.

26

$$\psi(t,p) = \frac{1}{\sqrt{1-2\|p\|^2 t}}\, p.$$

$$\psi : W \to \mathbb{R}^2$$

$$W = \{(t,p) \in \mathbb{R}\times\mathbb{R}^2 \mid 2\|p\|^2 t < 1\}$$

i) $I_p = \{t \mid t < \frac{1}{2}\frac{1}{\|p\|^2}\} = (-\infty, \frac{1}{2}\frac{1}{\|p\|^2})$ ✓ $p \neq (0,0)$

$I_{(0,0)} = \mathbb{R}$. ✓

iii) $\psi(0,p) = \frac{1}{\sqrt{1-0}}\, p = p.$

ii)

$$\begin{tikzcd} x \arrow[r, "s"] \arrow[rr, bend right, "\psi(s+t,\, x)"'] & \psi(s,x) \arrow[r, "t"] & \psi(t, \psi(s,x)) \end{tikzcd}$$

(b) Fix a point x_0 $\quad \gamma(t) = \psi(t, x_0)$ this is a curve

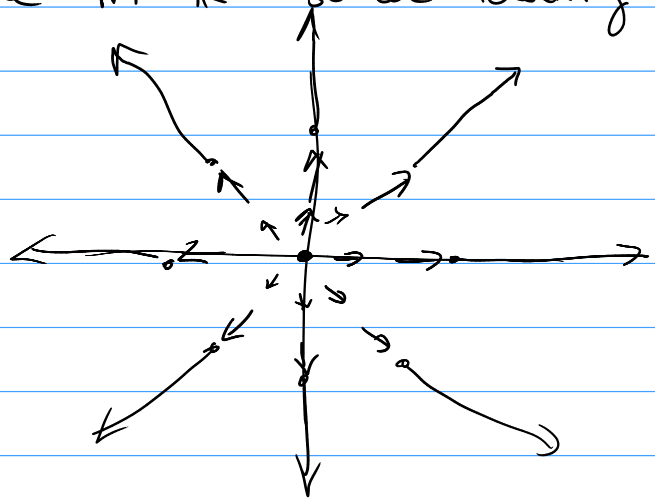
$t \in \mathbb{R} \longrightarrow \psi(t, x_0) \in X.$

$\gamma(0) = \psi(0, x_0) = x_0$

The vector at x_0 is just $[\gamma(t)]$

... but we are in $\mathbb{R}^2$ so we identify $[\gamma(t)] = \gamma'(0) = \frac{\partial}{\partial t}\psi\Big|_{t=0}$

$F(p) = \|p\|^2 p$



25(b)

$F(x,y,z) = (y,-x,0)$

Find the flow.

What's the differential equation $\gamma(t) = (x(t), y(t), z(t))$

$$\underbrace{[s \mapsto \gamma(t+s)]}_{\substack{\| \\ \gamma'(t)}} = \underbrace{F(\gamma(t))}_{\substack{\| \\ (y(t),-x(t),0)}}$$

in $\mathbb{R}^3$

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -x(t) \\ z'(t) = 0. \end{cases} \Rightarrow z(t) = z_0$$

$$x'' = y' = -x$$

$$x'' = -x$$

$x(t) = A\cos t + B\sin t$ $\qquad x(0) = x_0 = A$

$y(t) = x' = B\cos t - A\sin t.$ $\qquad y(0) = y_0 = B$

$$\gamma(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$ defined for $t \in (-\infty, \infty)$

$$\psi_F(t,p) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} p.$$

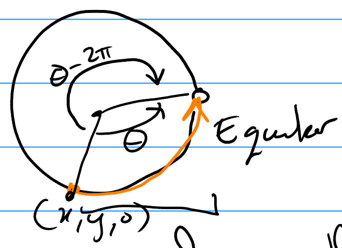
defined for $W = \mathbb{R} \times S^2 \Rightarrow$ global flow, maximal flow.

What is the flow / integral curve of the north pole. $N = (0,0,1)$

$$\psi_F(t,(0,0,1)) = \begin{pmatrix} & & \\ & & \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{for all } t.$$

$M = S^2 \setminus \{\overset{Q}{(1,0,0)}\}$

How can we define $W \subset \mathbb{R} \times M$ so that the flow is defined $\psi : W \to M$.



When does a point on the equator flow into Q. After time θ, or $\theta - 2\pi$.

$\theta, \theta - 2\pi > 1 - x$.

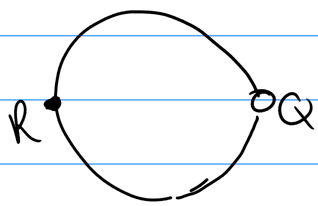
Its defined for $|t| < 1 - x$

$$W = \{t, p \mid t \in (-(1-x), 1-x)\}$$

Need to check $(s, \overset{(x_0,y_0,z_0)}{p}) \in W$ and $(t, \overset{(x_1,y_1,z_1)}{\psi(s,p)}) \in W$ then $(s+t, p) \in W$.

$|s| < 1 - x_0$ and $|t| < 1 - x_1$ $\quad x_1 = x_0 \cos s + y_0 \sin s$

$|s+t| < 1 - x_0$ TODO: Check this.



W says $I_R = (-2, 2)$