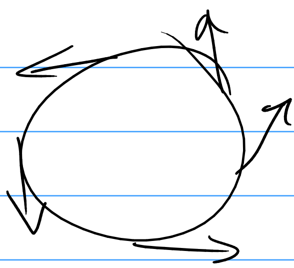


$F : S^1 \to TS^1$

$F(x) = (\tilde{F}(x), x)$

in $\mathbb{R}^2$

$\tilde{F}(x)\ (-x_2, x_1)$

As a derivation it is $\theta_F = \tilde{F}(x)\left(-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1}\right)$.

eg $\tilde{F}(x) = x_1^2 + x_2 \qquad g \in C^\infty(S^1, \mathbb{R}) \qquad g(x) = x_1 x_2$

F acts on g as a derivation to give the function

$\tilde{F}(x)\left(-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1}\right) g(x) = \underbrace{(x_1^2 + x_2)(-x_1 + x_2)} = \theta_F(g)$

$\theta_F(g) = 0$

What can we do now.

$\theta_F : \underline{C^\infty(X, \mathbb{R})} \to \underline{C^\infty(X, \mathbb{R})}$

$\theta_G(\theta_F(g)) \in C^\infty(X, \mathbb{R})$

Kor 2.3 $\quad g \mapsto \theta_F(\theta_G(g)) - \theta_G(\theta_F(g))$ is a derivation of smooth functions

$x^2 = x \cdot x$

$\frac{d^2}{dx^2} \downarrow \qquad \searrow \quad \underbrace{x \frac{d^2}{dx^2} x + \left(\frac{d^2}{dx^2} x\right) x}$

$2 \qquad\qquad x \cdot 0 + 0 \cdot x = 0.$

$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} g\right) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} g\right) = 0$

$\theta(fg) \quad f \cdot \theta(g) + g \cdot \theta(f)$

$\overset{\shortparallel}{0} \quad = \quad 0 + 0$

2 vectors field F, G $\longrightarrow$ 2 derivations θ_F, θ_G $\longrightarrow$ new derivation $\theta_F \circ \theta_G - \theta_G \circ \theta_F$ $\longrightarrow$ new vector field $[F, G]$

$v \in T_x X$ $\qquad \phi(x) = 0$

Directional derivative of $\quad w = (\phi \circ \alpha)'(0) \in \mathbb{R}^n$

$$D_v = D_{[\alpha]} = D_w(g)^{(x)} = \frac{d}{dt}\Big|_{t=0} g(\phi^{-1}(wt)) \qquad \text{general formula}$$

$$= \frac{d}{dt}\Big|_{t=0} g(x + wt) \qquad \text{in } \mathbb{R}^n$$

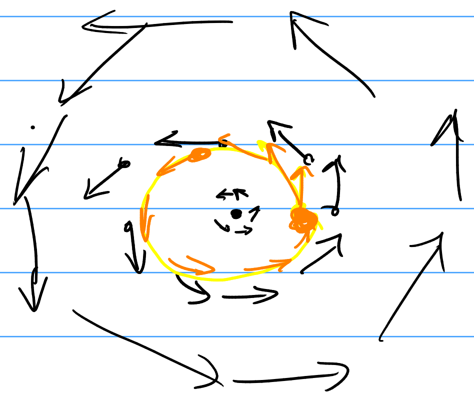
$$= \nabla g \cdot w \qquad \text{by chain rule}$$

$$= \nabla g(x) \cdot F(x)$$

do yourself for example F on S^1 above

$F : \mathbb{R}^2 \to \mathbb{R}^2$

$F(x) = (-x_2, x_1)$



the path of the duck is a circle. It is the integral curve of the vector field starting at some point.

The flow of a vector field is all of its integral curves.

integral curve eqn $\alpha(t)$ starting at x_0 is

$$\alpha'(t) = F(\alpha(t)) \qquad \text{in } \mathbb{R}^n$$

$$(\phi \circ \alpha)'(t) = T_{\alpha(t)}(\phi)\,(F(\alpha(t))) \qquad \text{in general.}$$

$$\begin{pmatrix} \alpha_1' \\ \alpha_2' \end{pmatrix} = \begin{pmatrix} -\alpha_2 \\ -\alpha_1 \end{pmatrix} \qquad \text{with} \quad \alpha(0) = x_0$$

$$\alpha_{x_0}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} = \begin{pmatrix} x_{02}\cos t + x_{01}\sin t \\ \ldots \end{pmatrix}$$

$$\phi(t, x) = \alpha_x(t)$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} x.$$