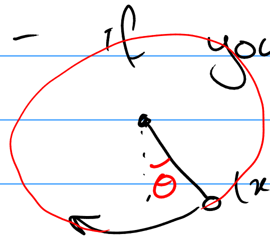
Tutorial 7

Why manifolds? — basis of (modern) geometry
— If you have parameters and constraints $\Rightarrow$ manifold



(v_x, v_y) 4 parameters 2 constraints

$x^2+y^2=1$ $x v_x + y v_y = 0$

$(\omega, \theta) \in TS^1$

Why abstract? — not all manifolds are obviously submanifolds
— Whitney embedding thm: all manifolds can be embedded in $\mathbb{R}^M$

Why vector bundles. — tangent bundle $\leftarrow$ vector fields
— dual tangent bundle $\leftarrow$ integration

Vector bundle X, E total manifold

B base manifold

π projection $: X \to B$ smooth surjective

$\pi^{-1}[b]$ is a vector space.

$B = S^1$

trivial bundle	tangent bundle	Möbius bundle
$X = \mathbb{R} \times S^1$	$X =$ vectors tangent to S^1 with their base point	$X = \mathbb{R} \times U_S$ glued to $\mathbb{R} \times U_N$
$\pi(r, b) = b$	$= \bigcup_{b \in S^1} T_b S^1$	$\pi(\text{point}) =$ the point in U_S or U_N $\subset S^1$
	$\pi(\text{vector}) =$ base point	

A section is a map from $s: B \to X$ with
$$\begin{tikzcd} X \arrow[d, "\pi"] \\ B \arrow[u, bend left, "s"] \end{tikzcd}$$
$\pi \circ s = id_B$.

for the UCB in the special cover we have

$$\begin{tikzcd} \pi^{-1}[U] \arrow[rr, "\psi", "\sim"'] \arrow[dr, "\pi"'] & & \mathbb{R}\times U \arrow[dl, "p_2"] \\ & U & \end{tikzcd}$$

trivial bundle

$U = S^1$

$\psi(r,b) = (r,b)$

$\underset{\pi^{-1}[U]}{\in} \qquad \underset{\mathbb{R}\times U}{\in}$

tangent bundle. $TS^1 = \{ (v,x) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \overset{S^1}{\|x\|=1} \quad v\circ x = 0 \} \subseteq \mathbb{R}^4$

Satz 1.54(i)

eg on $U = U_S = S^1 \setminus \{x_N\}$ $\qquad \phi_N(x_1,x_2) = \dfrac{x_1}{1-x_2}$

$$\pi^{-1}[U_S] = \{ (v,x) \in \mathbb{R}^2 \times U_S \mid v\circ x = 0 \}$$

why $\simeq \mathbb{R}\times U_S$?

The coord chart gives a trivialization because $\phi_N : U_S \to \mathbb{R}$

$$\begin{array}{ccc} T_x(\phi_N) : T_x U_S & \longrightarrow & T_{\phi_N(b)}\mathbb{R} \\ \| & & \| \\ \pi^{-1}[x] & \longrightarrow & \mathbb{R} \end{array}$$

$\psi : \pi^{-1}[U_S] \to \mathbb{R}\times U_S$

$$\psi(v,x) = (T_x(\phi_N)v, \pi(v,x)) = (T_x(\phi_N)v, x)$$

$$T_x(\phi_N)\, v = \left(\frac{1}{1-x_2}, \frac{x_1}{(1-x_2)^2} \right) \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{v_1(1-x_2) + x_1 v_2}{(1-x_2)^2}$$

$$\psi_{U_S}(v,x) = \Bigg(\underset{\mathbb{R}}{\underset{\uparrow}{\frac{(1-x_2)v_1 + x_1 v_2}{(1-x_2)^2}}}, \underset{U_S}{\underset{\uparrow}{x}} \Bigg) \in \mathbb{R}\times U_S$$

is an example of a local trivialisation of TS^1.

To try at home, show directly this is a diffeomorphism. or bijective

Why is it important? Because a "half-chart" of X

$$\underset{\substack{\text{open subset}\\ \text{of } X}}{\pi^{-1}[\mathcal{U}]} \xrightarrow[\sim]{\psi} \mathbb{R}^n \times \mathcal{U} \xrightarrow[\sim]{id\times\phi} \mathbb{R}^n \times \phi[\mathcal{U}] \subseteq \mathbb{R}^n \times \mathbb{R}^m$$

If $\mathcal{U}$ has a coordinate chart ϕ.

Local triviality of Möbius.

$M = \mathbb{R}\times\mathcal{U}_S \cup \mathbb{R}\times\mathcal{U}_N$ with relation

$(r_1, x_1)_S \sim (r_2, x_2)_N$ exactly when $x_1 = x_2$ as points of S^1

$$g_{\mathcal{U}_N\mathcal{U}_S}(x)\cdot r_2 = r_1$$

$g_{\mathcal{U}_N\mathcal{U}_S}(x) = \text{sign}(x_1)\cdot$, $B = S^1$

$\pi((r_1, x_1)_S) = x_1$ $\quad \pi((r_2, x_2)_N) = x_2$ $\quad$ π is well defined.

Trivialisation ψ_S over $\mathcal{U}_S \subset S^1$

$$\psi_S(\underset{\in M}{(r,x)_S}) = \underset{\in \mathbb{R}\times\mathcal{U}_S}{(r,x)}$$

Trivialisation ψ_N over $\mathcal{U}_N \subset S^1$

$$\psi_N(\underset{\in M}{(r,x)_N}) = \underset{\in \mathbb{R}\times\mathcal{U}_N}{(r,x)}$$

Think about the point

$w = (3, (-1/\sqrt{2}, 1/\sqrt{2}))_S$
$= (-3, (-1/\sqrt{2}, 1/\sqrt{2}))_N$

$\psi_S(w) = (3, (1/\sqrt{2}, 1/\sqrt{2})) \in \mathbb{R}\times\mathcal{U}_S$

$\psi_N(w) = (-3, (-1/\sqrt{2}, 1/\sqrt{2})) \in \mathbb{R}\times\mathcal{U}_N$

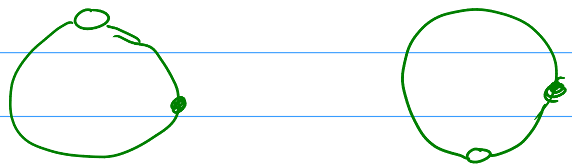
Do $\tilde{w} = (3, (1,0))_S$
$= (3, (1,0))_N$

$\psi_S((3,(1,0))) = (3, (1,0))$

$\psi_N(\tilde{w}) = (3, (1,0))$

Do $w_3 = (3, (0,1))_N$ $\quad \psi_N(w_3) = (3, (0,1))$

$(0,1)$ does not belong to $\mathcal{U}_S$ $\quad \cancel{\psi_S(w_3)} =$



U_S $\qquad$ U_N

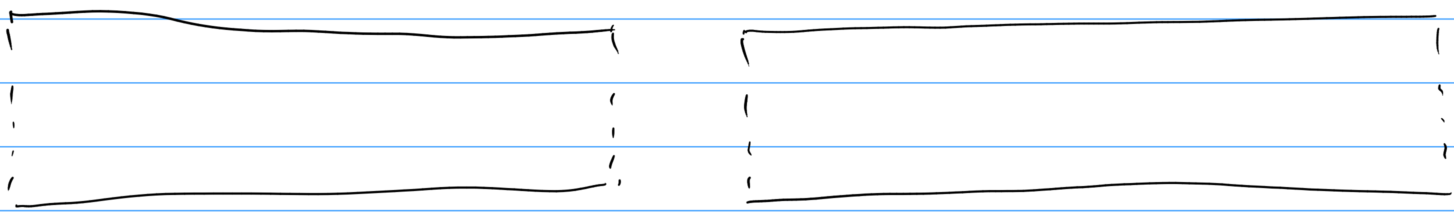
Sections of M. continuous function $S^1 \to M$ with $\pi \circ f = \mathrm{id}_{S^1}$

$$f(x) = \begin{cases} (x_1, x)_N & \text{if } x \in U_N \\ (|x_1|, x)_S & \text{if } x \in U_S \end{cases} \qquad x = (x_1, x_2) \in \mathbb{R}^2, \; x \in S^1.$$

well defined? if $x \in U_S \cap U_N$

$$(x_1, x)_N \sim ((\operatorname{sign} x_1) \cdot x_1, x)_S = (|x_1|, x)_S$$

$$\pi \circ f(x) = \begin{cases} \pi((x_1, x)_N) = x \\ \pi((|x_1|, x)_S) = x \end{cases}$$



$\mathbb{R} \times U_S$ $\qquad$ $\mathbb{R} \times U_N$

Draw f in the two trivializations and on your Möbius band.