Submanifold

Def 1.4 $f: X \to Y$ (a) immersion

$f: X \to f[X]$ is a homeomorphism

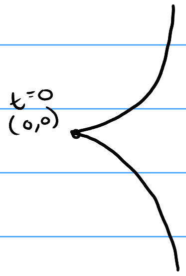
(b) $f: X \to f[X]$ is bijection

(c) f and f^{-1} are continuous

] embedding

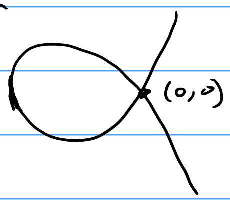
If f is an embedding, $f[X]$ is a submanifold

Ex1 bijective
Not immersion
$t \overset{\mathbb{R}\to\mathbb{R}^2}{\mapsto} (t^2, t^3)$



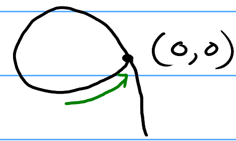
$T(f) = J(f) = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}$

Ex2 not injective
immersion
$t \mapsto \left(\frac{t^2-1}{t^2+1}, t\frac{t^2-1}{t^2+1}\right)$
$\mathbb{R} \to \mathbb{R}^2$



$f(1) = (0,0) = f(-1)$

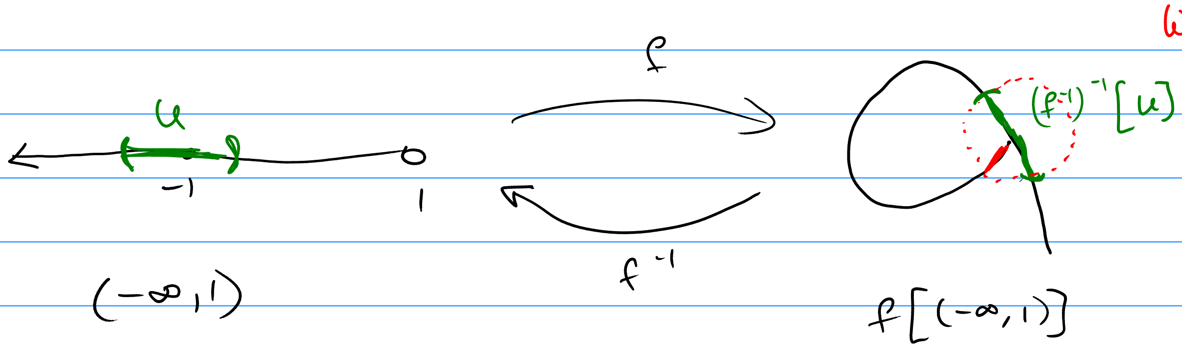
Ex3 injective, immersion
not homeomorphism
$t \mapsto \left(\frac{t^2-1}{t^2+1}, t\frac{t^2-1}{t^2+1}\right)$
$(-\infty, 1) \to \mathbb{R}^2$



$f^{-1}: f[(-\infty,1)] \to (-\infty,1)$ exist

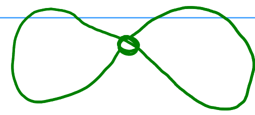
$f^{-1}(x,y) \mapsto \begin{cases} y/x & \text{for } x \neq 0 \\ -1 & \text{for } x = 0 \end{cases}$

$\lim f^{-1}(x,y) = 1$
$\neq f^{-1}(0,0) = -1$



What are the open sets of $f[X]$?
open sets V of Y $\cap f[X]$

f^{-1} is not continuous



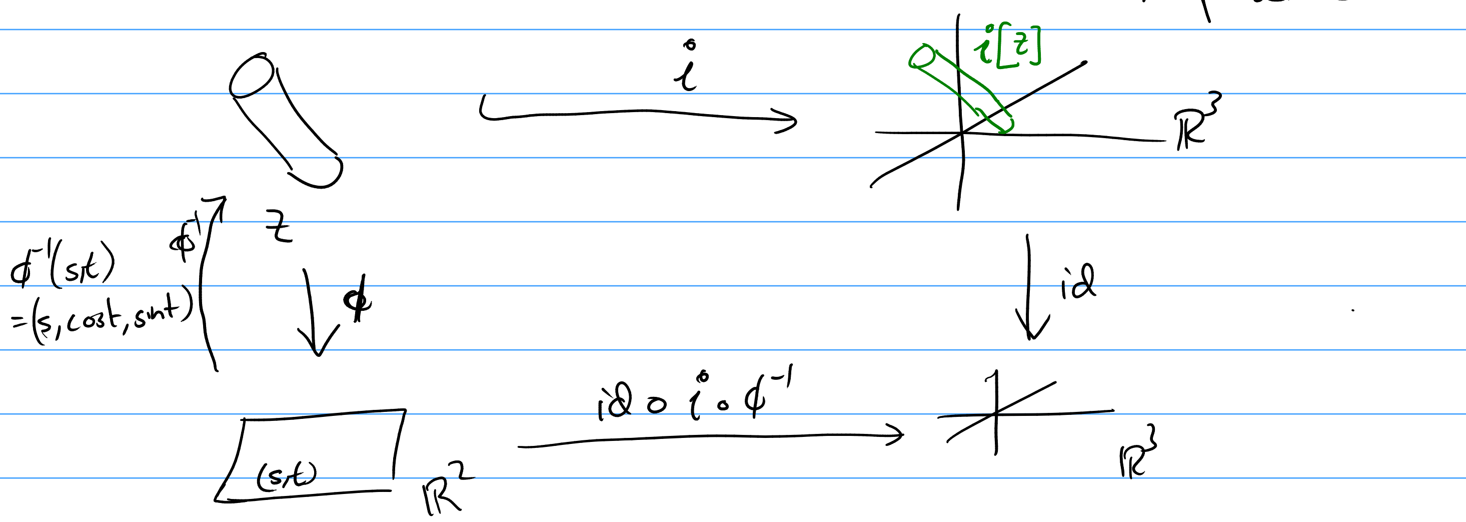
Q13(a) Show Z (the cylinder) is a submanifold of $\mathbb{R}^3$.
is a manifold

$Z = \{ x_2^2 + x_3^2 = 1 \} \subseteq \mathbb{R}^3$ is a <u>subset</u> of $\mathbb{R}^3$

Z is a <u>subspace</u> of $\mathbb{R}^3$ because it has the subspace topology.

Z is a manifold because it has an atlas, is it a <u>submanifold</u>

"Inclusion map" is the $i = id_{\mathbb{R}^3}|_Z : Z \to \mathbb{R}^3$ is a smooth map between manifolds



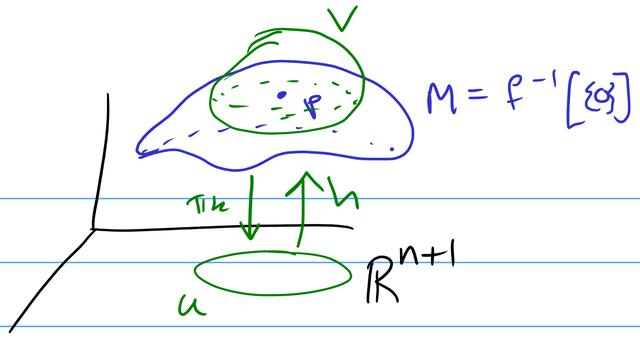
Z is a submanifold $\Leftrightarrow$ i is an embedding.

(a) immersion: $J(id \circ i \circ \phi^{-1}) = J(\phi^{-1}) = \begin{pmatrix} ① & 0 \\ 0 & -\sin t \\ 0 & \cos t \end{pmatrix}$ is rank 2 $= \dim Z$

(b) i is obviously injective because $i = id_{\mathbb{R}^3}|_Z$
all maps are surjective $f : X \to f[X]$

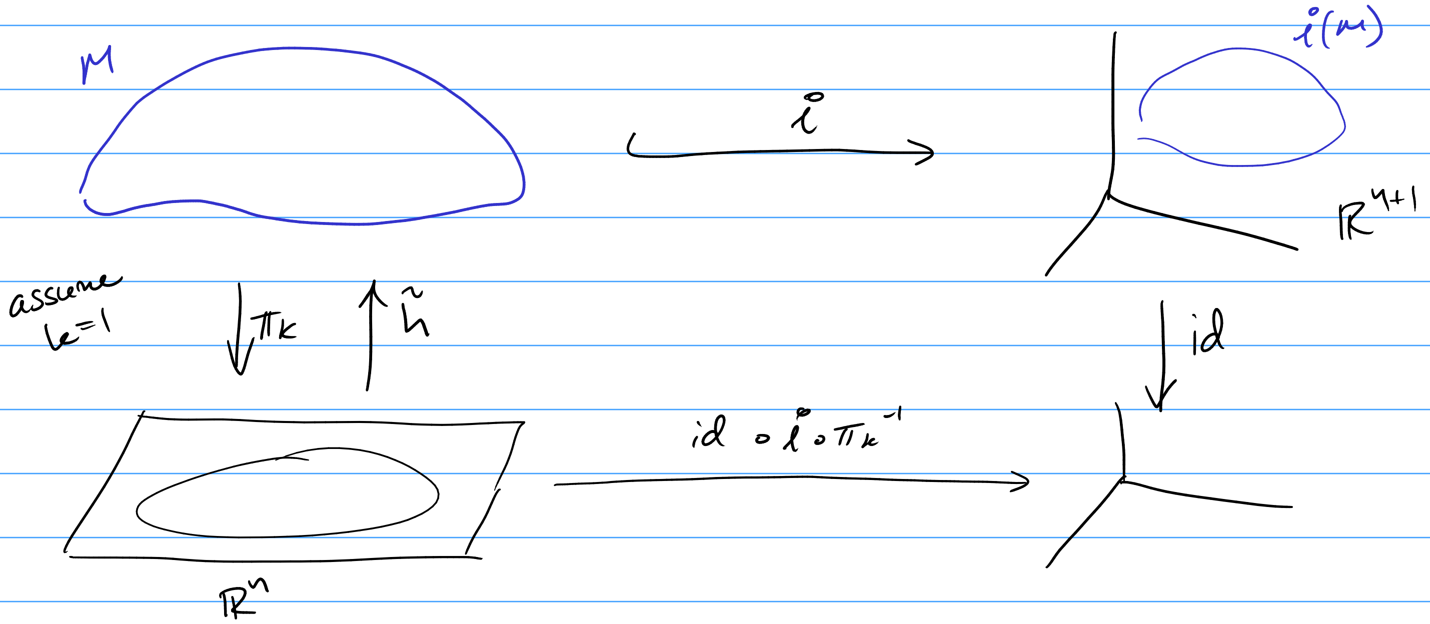
(c) i is continuous $id \circ i \circ \phi^{-1}(s,t) = (s, \cos t, \sin t)$
hard part i^{-1} is continuous.
$i^{-1} = (id_{\mathbb{R}^3}|_Z)^{-1} = id|_{Z \subset \mathbb{R}^3}$ is continuous.

13(b)



$f(x) = x_1^2 + x_2^2 + x_3^2 - 1 = 0$

Implicit function theorem says if $\nabla f = 0$ on M then at every point $p \in M$ $\exists\, V \subset \mathbb{R}^{n+1}$, a coordinate x_k, a height function $h(x_1, \dots, \hat{x}_k, \dots, x_{n+1})$ $V \cap M$ = graph of h.



(a) Immersion: $T(i) = J(id \circ i \circ \pi_k^{-1}) = J(\tilde{h})$

$$= J\; y \mapsto (h(y_1, \dots, y_n), y_1, y_2, \dots, y_n)$$

$$= \begin{pmatrix} \frac{\partial h}{\partial y_1} & \frac{\partial h}{\partial y_2} & \cdots & \frac{\partial h}{\partial y_n} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ & & & \\ 0 & 0 & \cdot & 1 \end{pmatrix} = \begin{pmatrix} \\ I_n \end{pmatrix}$$

rank $J = n = \dim M$

b) clear: always a bijection for inclusion

c) i is continuous because M has the subspace topology so i is continuous. See solutions

~~i^{-1} is the restriction of a continuous map $id_{\mathbb{R}^{n+1}}|_Z$~~

Sphere $f(x,y,z) = x^2+y^2+z^2 - r^2$ $\quad r>0$

17a) $f(x,y,z) = \left(a - \sqrt{x^2+y^2}\right)^2 + z^2 - b^2$

$T = f^{-1}[\{0\}]$

Summarize 13(b) as $\nabla f \neq 0 \Rightarrow M$ a manifold ex 1.18(iv)

$\Rightarrow M$ is a submanifold with the those charts.

$$\partial_x f = 2\left(a - \sqrt{x^2+y^2}\right)\cdot\left(-\cancel{2}\frac{1}{\sqrt{x^2+y^2}}\cancel{2}x\right)$$

$$= \frac{2x}{\sqrt{x^2+y^2}}\left(a - \sqrt{x^2+y^2}\right)$$

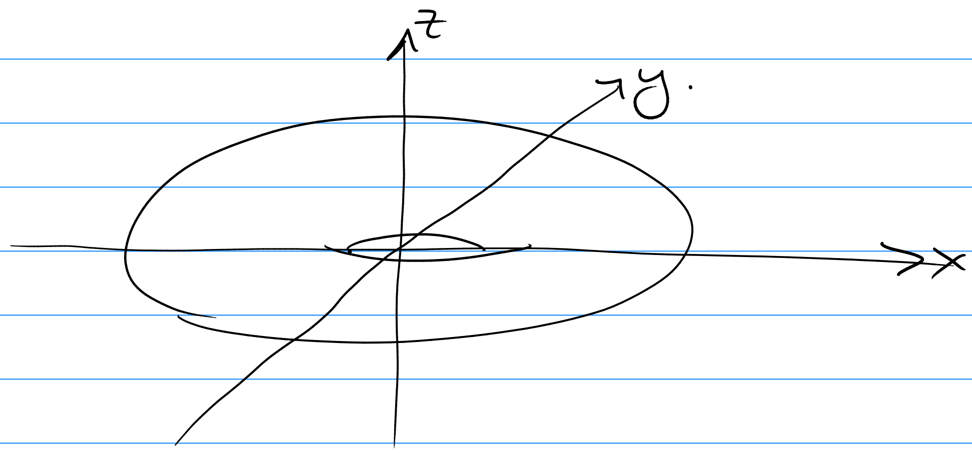
$$\partial_y f = \frac{2y}{\sqrt{x^2+y^2}}\left(a - \sqrt{x^2+y^2}\right) \qquad \partial_z f = 2z.$$

Is $\nabla f = 0$ on M? $\quad 0<b<a$

$z=0 \Rightarrow a - \sqrt{x^2+y^2} = \pm b \neq 0$

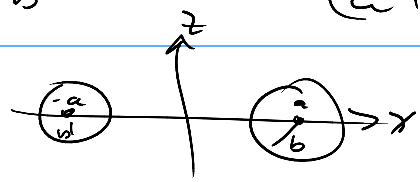
$(a\pm b)^2 = x^2+y^2 \neq 0$

$\Rightarrow x=y=0 \Rightarrow a = \pm b$ contradiction.



set $y=0$ $\qquad (a-|x|)^2 + z^2 = b^2$

$(a-x)^2+z^2=b^2 \qquad (a+x)^2+z^2=b^2$



$t \in \mathbb{R}$

b). $A_t = \{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = t \}$

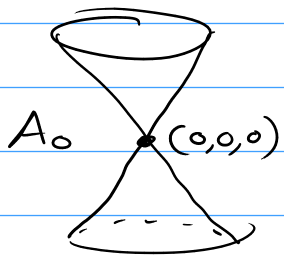
$A_t = \{ f_t(x) = x_1^2 + x_2^2 - x_3^2 - t = 0 \}$

$\nabla f = (2x_1 , 2x_2 , -2x_3)$

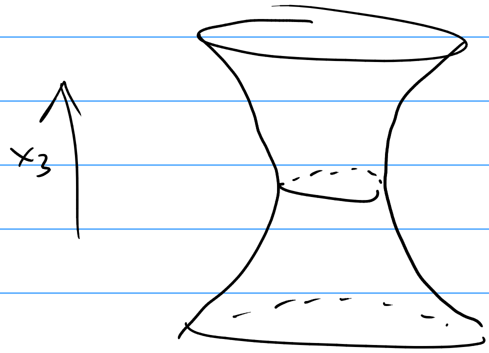
$\nabla f = 0 \Leftrightarrow x = (0,0,0) \in A_0$

$t \neq 0$ A_t is a submanifold

$t = 0$ A_t is not a submanifold

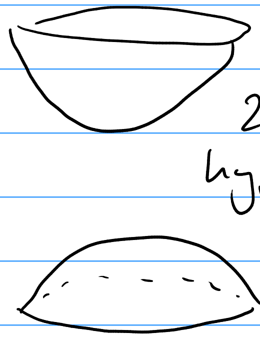


$t_1 = 1.$ $x_1^2 + x_2^2 = 1 + x_3^2$



1 sheeted hyperboloid

$t = -1$



2 sheeted hyperboloid.