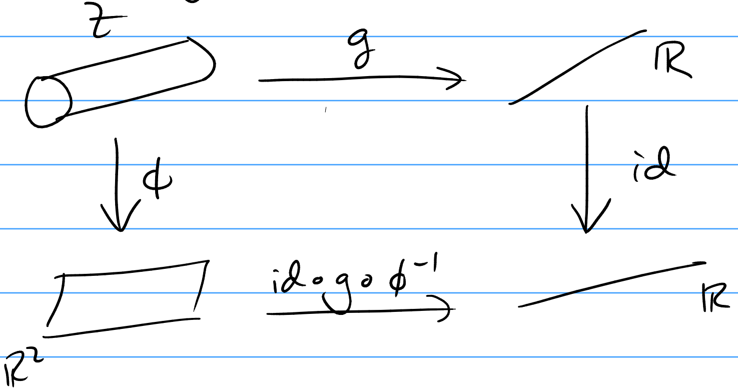
Tutorial 4.

Recall definition of a smooth function.

$Z = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2^2 + x_3^2 = 1\}$

$g: Z \to \mathbb{R}$ $\quad g(x_1, x_2, x_3) = x_1 + \sqrt{|x_2^2 + x_3^2 - 1|}$



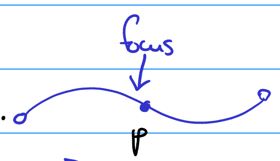
g's smoothness as a function $\mathbb{R}^3 \to \mathbb{R}$ is irrelevant.

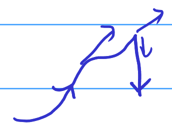
Z is <u>not</u> a vector space.

We want to describe directions in the manifold without referring to an outside $\mathbb{R}^N$.

a vector at p is an equivalence

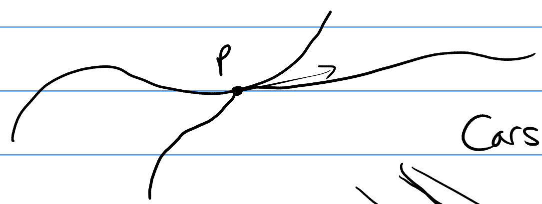
Idea: talk about classes of smooth paths.



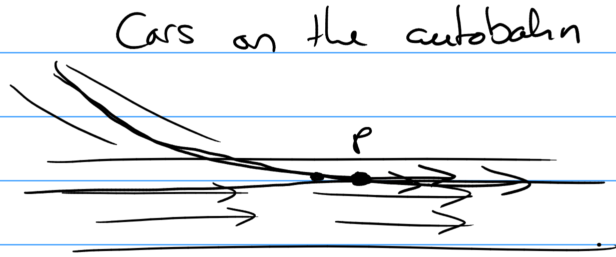


Recall: a path is a cts $(-\varepsilon, \varepsilon) \to Z$

3. Two paths maybe different, but travelling in same direction at p.



1. Directions in Z.
2. Look at paths

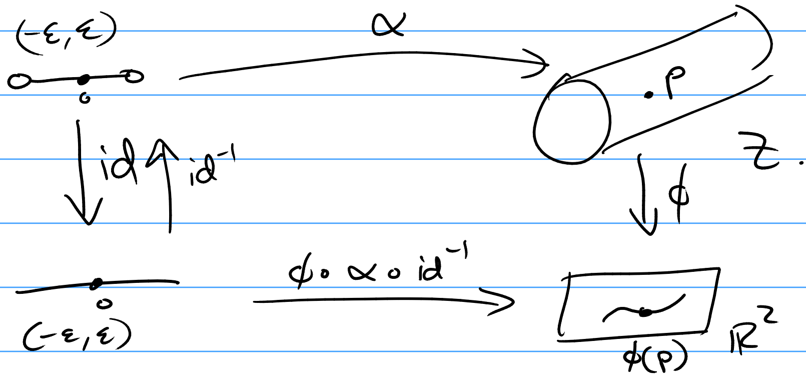


"a path <u>through</u> p" $\quad \alpha: (-\varepsilon, \varepsilon) \to Z$ smooth, $\alpha(0) = p$.

How do we determine if two paths have the same direction at p?

Use a chart

Def 1.32.



Now in euclidean space, can compare tangent vector of $(\phi\circ\alpha)$

α and β are tangent at p if $(\phi\circ\alpha)'(0) = (\phi\circ\beta)'(0)$

$\alpha,\beta : (-\varepsilon,\varepsilon) \to Z$ $\quad \alpha(0)=\beta(0)= p = (0,1,0)$

$\alpha(\lambda) = (0, \cos\lambda, \sin\lambda)$ $\qquad \beta(\lambda) = (\lambda^2, \sqrt{1-\lambda^2}, \lambda)$

$\phi = f_{-\pi}(x_1, \arcsin x_3)$

$(\phi\circ\alpha)(\lambda) = \phi(\overset{x_1}{0}, \overset{x_2}{\cos\lambda}, \overset{x_3}{\sin\lambda})$

$= (0, \lambda)$

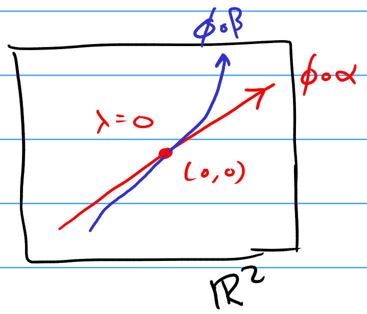
$(\phi\circ\alpha)'(0) = (0,1)$

$(\phi\circ\beta)(\lambda) = \phi(\overset{x_1}{\lambda^2}, \overset{x_2}{\sqrt{1-\lambda^2}}, \overset{x_3}{\lambda})$

$= (\lambda^2, \arcsin\lambda)$

$(\phi\circ\beta)'(\lambda) = (2\lambda, 1/\sqrt{1-\lambda^2})$

$(\phi\circ\beta)'(0) = (0, 1)$



$\psi(x_1, x_2, x_3) = (x_3, x_1)$

$(\psi\circ\alpha)(\lambda) = (\sin\lambda, 0)$

$(\psi\circ\alpha)'(\lambda) = (\cos\lambda, 0)$

$(\psi\circ\alpha)'(0) = (1, 0)$

$(\psi\circ\beta)(\lambda) = (\lambda, \lambda^2)$

$(\psi\circ\beta)'(\lambda) = (1, 2\lambda)$

$(\psi\circ\beta)'(0) = (1, 0)$

$\alpha \sim \beta \iff$

Assume $(\phi\circ\alpha)'(0) = (\phi\circ\beta)'(0)$

Need to show $(\psi\circ\alpha)'(0) = (\psi\circ\beta)'(0)$

$$(\psi\circ\alpha)'(0) = \Big(\underbrace{(\psi\circ\phi^{-1})}_{\mathbb{R}^n\to\mathbb{R}^n}\circ\underbrace{(\phi\circ\alpha)}_{\mathbb{R}\to\mathbb{R}^n}\Big)'(0)$$
$$= (\psi\circ\phi^{-1})'(pt)\cdot(\phi\circ\alpha)'(0)$$
$$= (\psi\circ\phi^{-1})'(pt)\cdot(\phi\circ\beta)'(0)$$
$$= (\psi\circ\phi^{-1}\circ\phi\circ\beta)'(0)$$
$$= (\psi\circ\beta)'(0).$$

Proof that tangency is independent of chart.

$[\alpha]$ is a tangent vector to Z at p.

T_pZ is all vectors at p $\qquad 0 = \begin{bmatrix}\text{constant map}\\ \alpha(\lambda) = p\end{bmatrix}$

$$\gamma(\lambda) = (-\lambda^2, \cos 2\lambda, \sin 2\lambda)$$

$$(\phi\circ\gamma)(\lambda) = (-\lambda^2, 2\lambda)$$
$$(\phi\circ\gamma)'(0) = (0, 2) \qquad [\alpha]+[\beta] = 2[\alpha] = [\gamma]$$

Tangent Map $T_p(f)$ at p.

a linear map : $T_pX \to T_{f(p)}Y$

$$(-\epsilon,\epsilon) \xrightarrow{\alpha} X \xrightarrow{f} Y$$

input is a path α through p.

output is a path through $f(p)$, namely $\beta = f\circ\alpha$

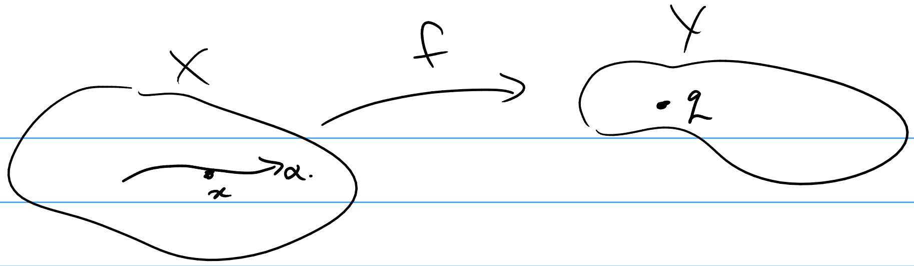
$$\beta(0) = f(\alpha(0)) = f(p)$$

- Linear
- Doesn't depend on the choice of path in equivalence class.

$\alpha \sim \tilde{\alpha} \qquad f\circ\alpha \sim f\circ\tilde{\alpha}$

So $T_p(f)([\alpha]) = [f\circ\alpha]$ is well-defined.

e)



Want to prove $T_x(f)([\alpha]) = 0 \iff T_x(f) = 0$

$$T_x(f)([\alpha]) = [f \circ \alpha] = [\lambda \mapsto q] = 0$$

Immersion

$$\begin{array}{ccc} X = \mathbb{R}^n & \xrightarrow{f} & Y = \mathbb{R}^m \\ \downarrow \scriptstyle{id} & & \downarrow \scriptstyle{id} \\ \square & \xrightarrow[\substack{id \circ f \circ id^{-1} \\ = f.}]{} & \square \end{array}$$

Vectors α, β in X

$$(id \circ \alpha)'(0) = (id \circ \beta)'(0)$$
$$\alpha'(0) = \beta'(0)$$

$$\underline{T_p(f)}([\alpha])$$
$$= [f \circ \alpha]$$
$$\approx (f \circ \alpha)'(0)$$
$$= f'(\alpha(0)) \cdot \alpha'(0)$$
$$= \underline{\underline{f'}} \cdot [\alpha]$$

For Euclidean space Tangent map $\equiv$ Jacobian.

A map f is an immersion at p if T_pf is injective as a linear map $T_pX \to T_{f(p)}Y$

(a) i) $f: \mathbb{R}^1 \to \mathbb{R}^3$ $\quad f(t) = (\cos 2t, \sin 2t, t)$

$$f'v = \begin{pmatrix} -2\sin 2t \\ 2\cos 2t \\ 1 \end{pmatrix} v = \begin{pmatrix} (-2\sin 2t)\, v \\ (2\cos 2t)\, v \\ v \end{pmatrix} \qquad v \in \mathbb{R}$$

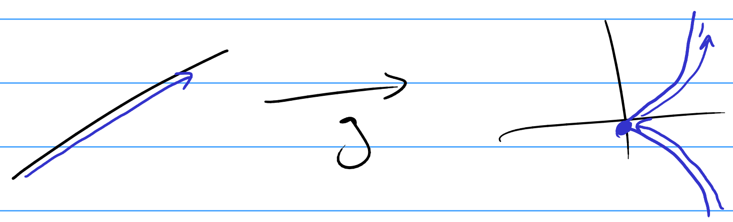
for all t $\quad f'$ is injective function of v.

ii) $g: \mathbb{R} \to \mathbb{R}^2$ $\quad g(t) = (t^2, t^3)$ $\qquad g' = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}$

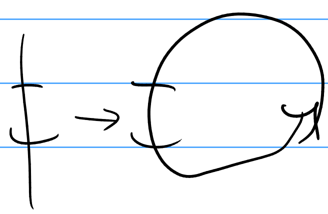
$$g'v = \begin{pmatrix} 2t\,v \\ 3t^2 v \end{pmatrix}$$

$\nearrow t \neq 0$ injective

$\searrow t = 0$ not injective



iii) $h: \mathbb{R} \to \mathbb{R}^2$ $\quad t \mapsto (\cos t, \sin t)$



immersion. but not an injection