$\varphi : U \to \mathbb{R}^n$ homeomorphism onto image

$\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}^1$

manifold $\quad\quad$ $\mathbb{R}^n$

Q7

$$\chi(x) = \begin{cases} x & \text{if } x \le 0 \\ 2x & \text{if } x > 0 \end{cases}$$ is a chart of $\mathbb{R}$

$\chi : \mathbb{R} \to \mathbb{R}$

top space $\quad$ euclidean

χ is a bijection $\quad \chi^{-1}(x) = \begin{cases} x & \text{if } x \le 0 \\ \frac{1}{2}x & \text{if } x > 0 \end{cases}$

By analysis I χ and χ^{-1} are continuous. χ is a chart.

$\tilde{A} = \{\chi\}$ is an atlas. $\quad (\mathbb{R}, A)$ "normal" $\mathbb{R}$ as a manifold

$(\mathbb{R}, \tilde{A})$ "weird" $\mathbb{R}$, $\tilde{\mathbb{R}}$

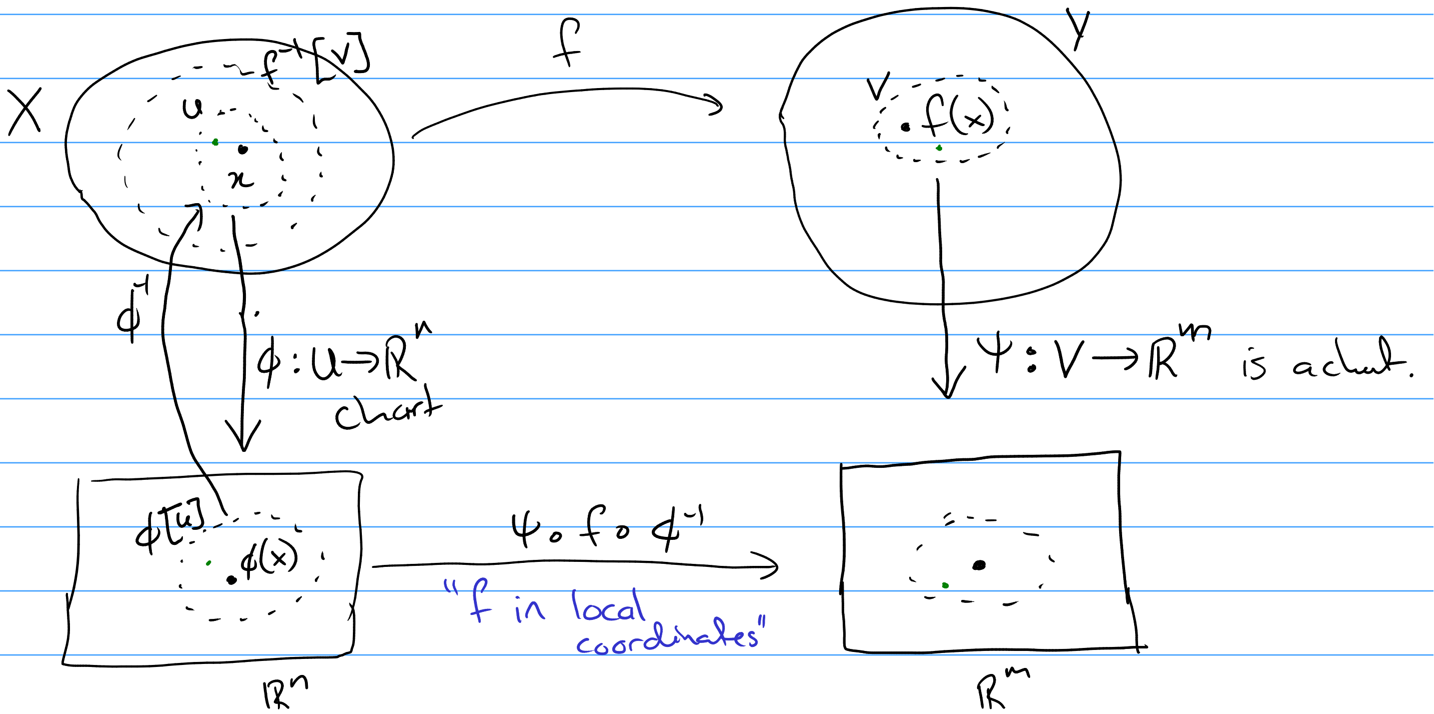
φ and ϕ are compatible when
- $\varphi \circ \phi^{-1}$ smooth
- $\phi \circ \varphi^{-1}$ smooth

transition function

$\chi \circ \mathrm{id}^{-1}(x) = \chi(x)$ not smooth.

glatt.

Smooth functions between manifolds.



Normal $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ ← vector space operation

We say f is smooth at x (in the manifold sense) $\Leftrightarrow$
$\psi \circ f \circ \phi^{-1}$ is smooth at $\phi(x)$ (in the euclidean sense)

8a) $X = \mathbb{R}^n$, $\phi = id_{\mathbb{R}^n}$ $\xrightarrow{F}$ $Y = \mathbb{R}^m$, $\psi = id_{\mathbb{R}^m}$

F is manifold-smooth at a

$\Leftrightarrow$ $\psi \circ F \circ \phi^{-1}$ is euclidean-smooth at $\phi(a)$

$\Leftrightarrow$ $id_{\mathbb{R}^m} \circ F \circ id_{\mathbb{R}^n}^{-1}$ is euclidean smooth at $id_{\mathbb{R}^n}(a)$

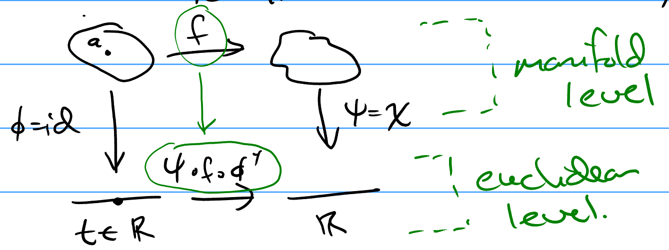
$\Leftrightarrow$ F is euclidean-smooth at a.

From question 8a we know $F(x)=x^2$ is manifold smooth function from $\mathbb{R}\to\mathbb{R}$

chart

$\chi=\begin{cases} x, & x\le 0 \\ 2x, & x>0 \end{cases}$

Question: is F a smooth map from $\mathbb{R}$ to $\tilde{\mathbb{R}}$?



$\chi\circ F\circ id^{-1}(t) = \chi\circ F(t)$

$=\chi(t^2) = 2t^2$ this is smooth for at t.

all points of $\mathbb{R}$.

$G=id:\mathbb{R}\to\tilde{\mathbb{R}}$ is not smooth at $x=0$ because

$\chi\circ id\circ id^{-1}(t) = \chi(t)$ is not smooth at $x=0$

8c) $S^2 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0^2 + x_1^2 + x_2^2 = 1\} \subseteq \mathbb{R}^3$

topology = subspace topology from $\mathbb{R}^3$

Atlas $= \{N, S\}$

$: \mathbb{R}^3 \to \mathbb{R}^3$ cts because cts in each component

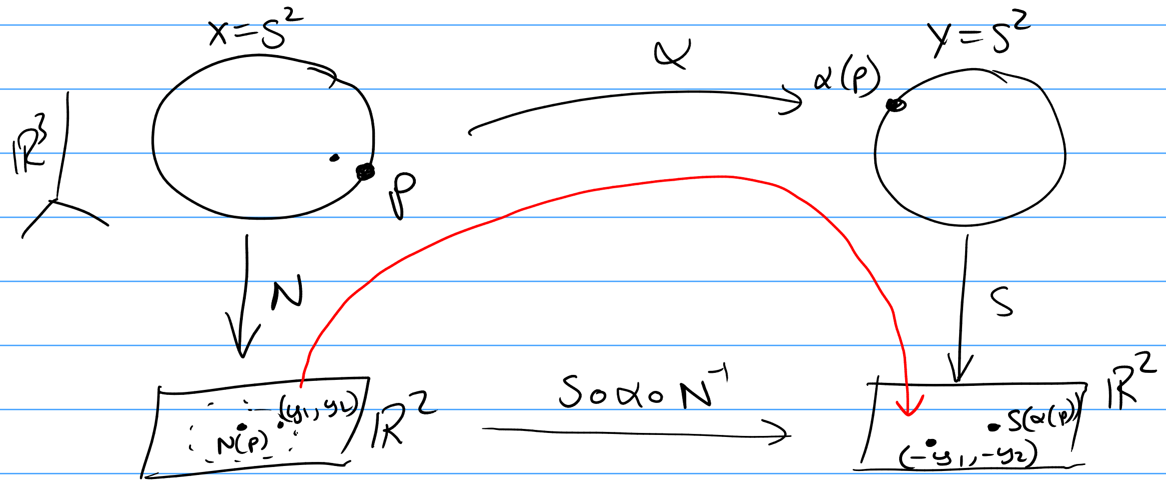
$\alpha(x_0, x_1, x_2) = (-x_0, -x_1, -x_2) \quad : S^2 \to S^2$

α is continuous because restriction to a subspace of a continuous function.

Want to show α is smooth at every point $p \in S^2$

Case 1. $p \neq (1,0,0)$ so we can use the chart N on a n'hood of p

$\alpha(p) \neq (-1,0,0)$ so use chart S.



$$S \circ \alpha \circ N^{-1}(y_1, y_2) = S \circ \alpha \left(\frac{(y_1^2 + y_2^2 - 1, 2y_1, 2y_2)}{\|y\|^2 + 1} \right)$$

$$= S\left(\underbrace{\frac{(1 - y_1^2 - y_2^2, -2y_1, -2y_2)}{\|y\|^2 + 1}}_{q} \right)$$

$$= -(1,0,0) + \frac{q + (1,0,0)}{1 + \langle q, (1,0,0) \rangle}$$

$$= \frac{\frac{(0, -2y_1, -2y_2)}{\|y\|^2+1}}{1 + \frac{1 - y_1^2 - y_2^2}{\|y\|^2+1}} = \frac{(0, -2y_1, -2y_2)}{1 + \cancel{y_1^2} + \cancel{y_2^2} + 1 - \cancel{y_1^2} - \cancel{y_2^2}}$$

$$= (-y_1, -y_2) \in \mathbb{R}^2$$

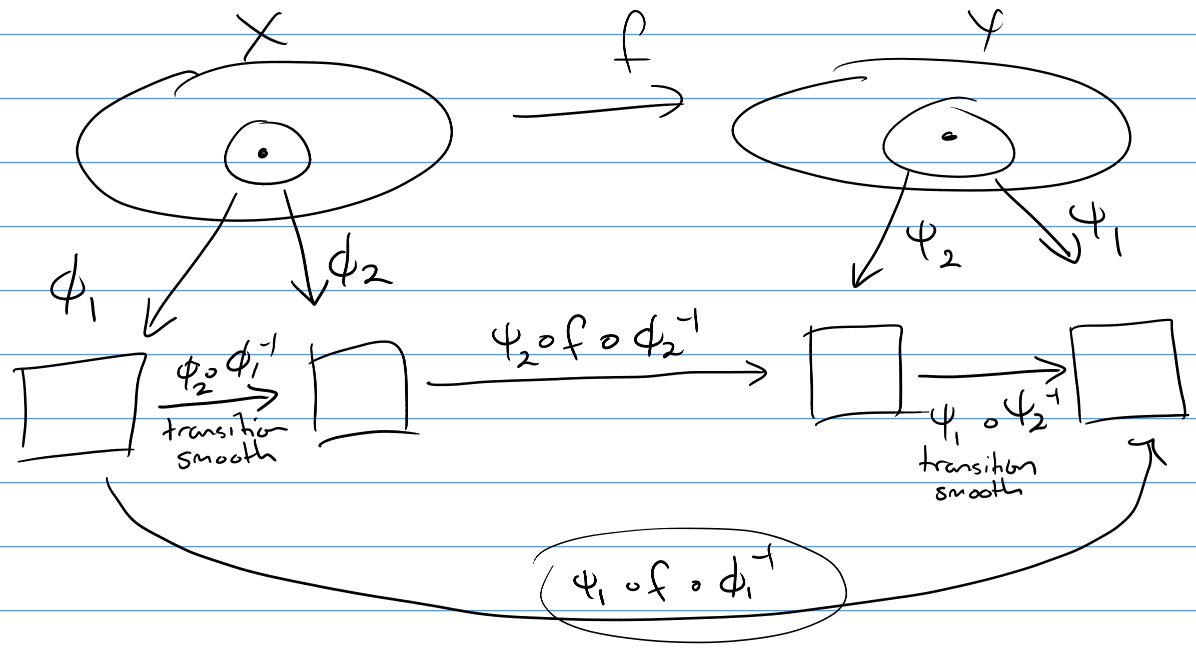
$(S\circ\alpha\circ N^{-1})(y_1, y_2) = (-y_1, -y_2)$ is a smooth function $\mathbb{R}^2 \to \mathbb{R}^2$

Next tutorial: Is it true '$X \subset \mathbb{R}^n \xrightarrow{F \text{ smooth}} Y \subset \mathbb{R}^m$'

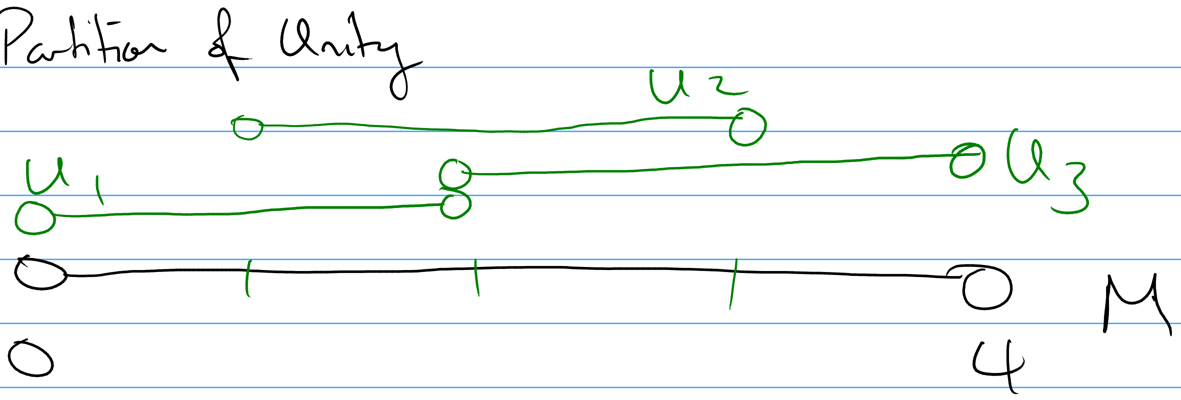
is $F|_X : X \to Y$ smooth in sense of manifolds

try it yourself.

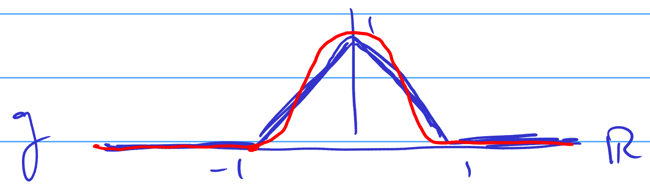
Why does the definition of smooth $X \to Y$ not depend on the particular chart in the atlas?



Partition of Unity

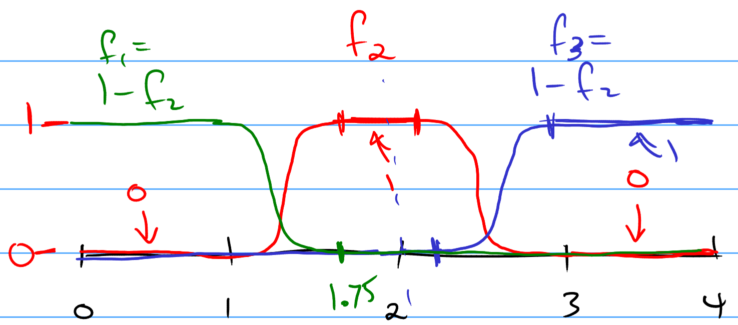


The support of a function: closure of the nonzero domain



$$\overline{\{x \mid g(x) \neq 0\}} = \overline{(-1,1)}$$

$$= [-1,1]$$



$$\text{supp } f_1 = \overline{(0, 1.75)} \text{ in } M$$

$$= (0, 1.75]$$

not compact.