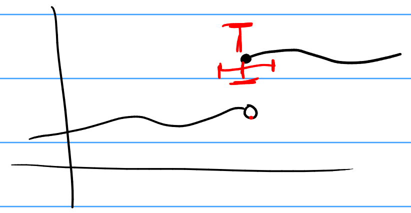
NB Neighbourhood of x. is any set
- $x \in N$
- the exist an open set U with $x \in U$ $U \subseteq N$.

open n'hood = open set containing x
closed n'hood
compact n'hood

eg in $\mathbb{R}$, x. $N = \{x\}$ is not a closed n'hood.

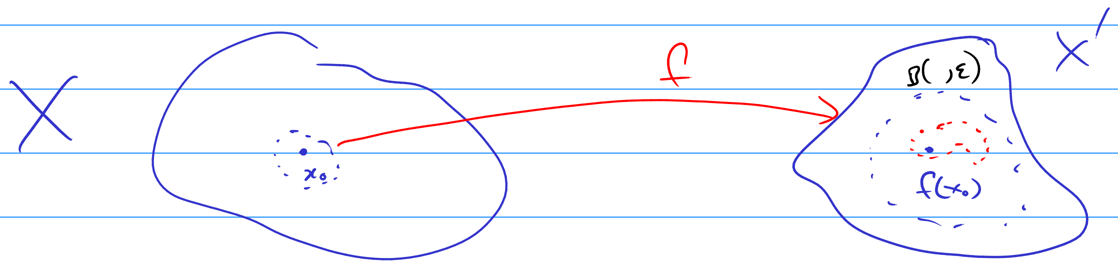
Q1. Basic definition of continuity is $f: \mathbb{R} \to \mathbb{R}$



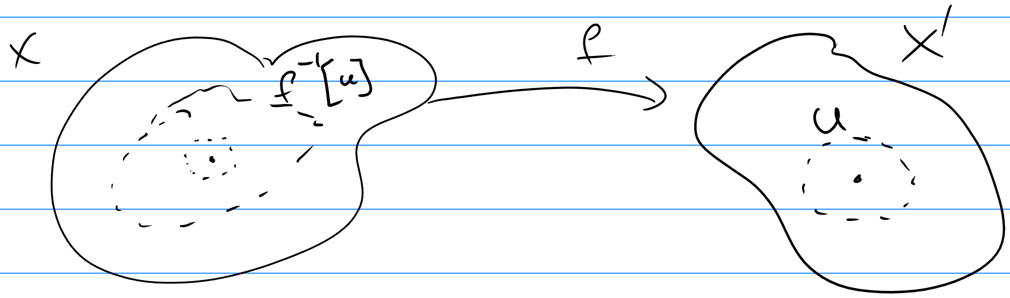
cts at x_0 $\forall \varepsilon > 0\ \exists \delta > 0$
$\forall x \in \{ |x - x_0| < \delta \}$
$\Rightarrow |f(x) - f(x_0)| < \varepsilon$

In metric space (2) cts at x_0 $\forall \varepsilon > 0\ \exists \delta$
$\underbrace{\forall x \in \{ d(x, x_0) < \delta \}}_{\forall x \in B(x_0, \delta)} \Rightarrow \underbrace{d'(f(x), f(x_0)) < \varepsilon}_{f(x) \in B(f(x_0), \varepsilon)}$

(2') $f[B(x_0, \delta)] \subseteq B(f(x_0), \varepsilon)$



(1') $\forall$ open sets U there is a open set in $f^{-1}[U]$



(1) $\forall$ open sets U , $f^{-1}[U]$ is open.

$(2) \Rightarrow (1)$ is the motivation for generalisation.

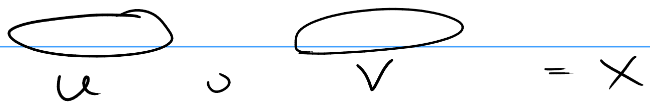
$(1) \Rightarrow (2)$ $\quad \forall V$ in X', open $\quad$ the preimage $f^{-1}[V]$ is open.

Choose any point $x_0 \in X$, any $\varepsilon > 0$ $\quad$ consider $B(f(x_0), \varepsilon) = V$
then $f^{-1}[B(f(x_0), \varepsilon)]$ is open, contains x_0
the definition of open for a metric space is that it contains an open ball. $\exists\, \delta > 0$ $\quad B(x_0, \delta) \subseteq f^{-1}[B(f(x_0), \varepsilon)]$.

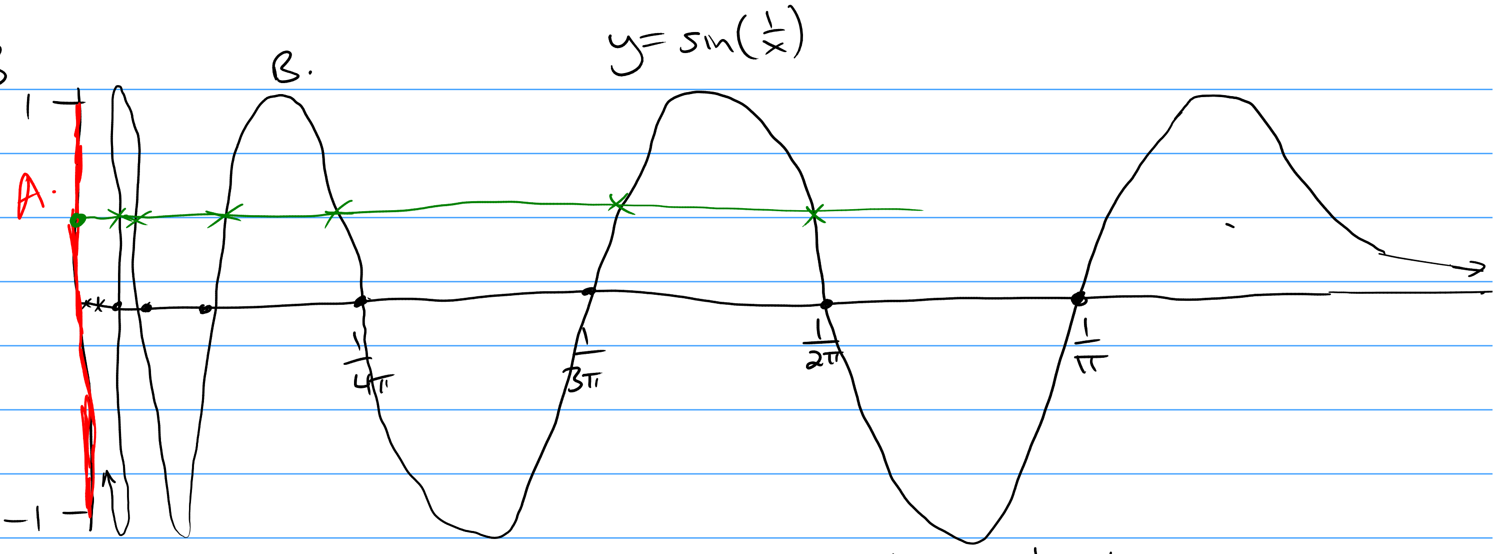
<u>Q2.</u> Two definition of disconnected = not connected

- Def 1.8 $\quad X$ is disconnect $\iff \exists\, Y \subset X$ that is open and closed.

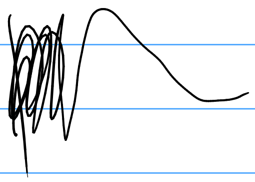
- X is disconnected $\iff \exists\, U, V$ open, nonempty so that
$U \cup V = X$, $\quad U \cap V = \emptyset$.



Q3



$0 = y = \sin(\frac{1}{x}) \Leftrightarrow \frac{1}{x} = \pi n \Leftrightarrow x = \frac{1}{\pi n} = \frac{1}{\pi} \times \frac{1}{n}$

Cartoon $\Rightarrow$ 

$(0,\infty)$ is connected, $f(x) = \sin(\frac{1}{x})$ is continuous, $x > 0$

$B = f[(0,\infty)]$ is connected.

b) If $\overline{B} = M$ then $B \subseteq M \subseteq \overline{B} \Rightarrow M$ is connected.

Take a sequence (x_n, y_n) in B that converges to $(\tilde{x}, \tilde{y})$

$y_n = \sin(\frac{1}{x_n})$

Because we use the normal/common metric space limits

$x_n \to \tilde{x} \qquad y_n \to \tilde{y}$.

Case 1 $\tilde{x} > 0$. $\tilde{y} = \lim y_n = \lim_{n\to\infty} \sin(\frac{1}{x_n}) = \sin(\frac{1}{\tilde{x}})$

$(\tilde{x}, \tilde{y}) \in B$.

case 2 $\tilde{x} \leq 0$ But we know $x_n > 0$ by comparison law

$\tilde{x} = 0 \qquad \tilde{x} = \lim x_n \geq \lim 0 = 0$

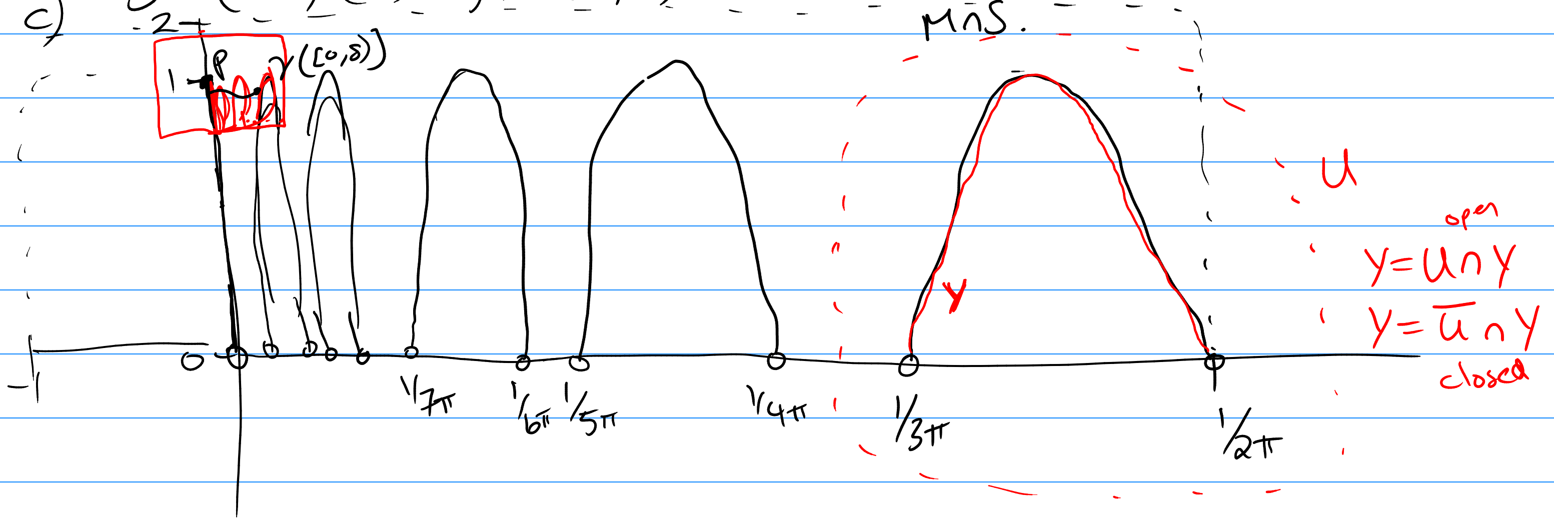
$-1 \leq y_n \leq 1$

$-1 \leq \tilde{y} \leq 1$ so $(\tilde{x}, \tilde{y}) \in A$.

so $\overline{B} \subseteq B \cup A$

Need to show $A \subseteq \overline{B}$

c) $S = (-1, (2\pi)^{-1}) \times (0,2)$



$X = M \cap S$ Why is Y open and closed in X?

Open sets of M are $U \cap M$ for U open in $\mathbb{R}^2$

open sets of $M \cap S$ are $(U \cap M) \cap (M \cap S) = U \cap (M \cap S)$

d) See red n'hood of p.

e) $pr_1(x,y) = x$ is a cts map $\mathbb{R}^2 \to \mathbb{R}$.

$\gamma(t) = (\gamma_1(t), \gamma_2(t))$

$\gamma_1(t) = (pr_1 \circ \gamma)(t)$ is cts function.

This says $\gamma_1[[0,1]]$ must be connected in $\mathbb{R}$, compact.

it is an closed interval

Assume that $\gamma(0) = p$ and $\gamma(t) \in B$ for $t > 0$.

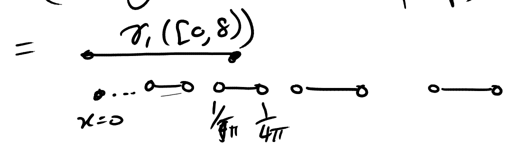
Because $\gamma_1(0) = 0$ $\gamma_1(t) > 0$ for $t > 0$

$\forall \varepsilon > 0 \ \exists \delta$ so that $\forall t \in [0,\delta) \Rightarrow |\gamma_1(t) - \gamma_1(0)| < \varepsilon$

$\gamma_1(t) \in (0, \varepsilon)$

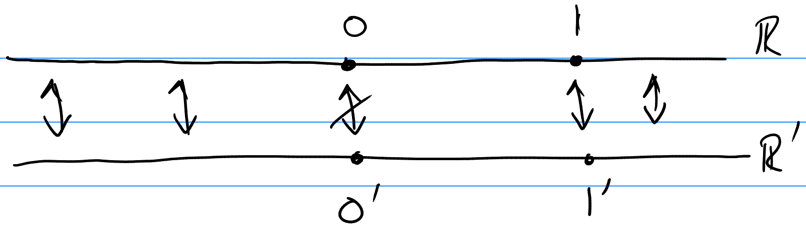
connected.

$\gamma_1([0,\delta)) \subseteq pr_1(\text{neighbourhood of } p)$



Contradiction

4a.



$x \sim x'$ if $x \in \mathbb{R}$ $x' \in \mathbb{R}'$

$x = x'$ except $0 \not\sim 0'$

$1 \sim 1'$ $\quad 2 \sim 2'$ $\quad (-1) \sim (-1)'$

$D = \mathbb{R} \sqcup \mathbb{R}' / \sim$ has point $\mathbb{R} \setminus \{0\}$, 0, $0'$

$\mathcal{U}$ is open in D if
- $\mathcal{U} \subset \mathbb{R}$ and open
- $0' \in \mathcal{U}$ and $\mathcal{U} \setminus \{0'\} \cup \{0\}$ is open in $\mathbb{R}$

(n^{-1}) converges to 0 and $0'$

Take any $\mathcal{U} \ni 0$, open. it contains $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$

But the $\exists N$ with $N > \frac{1}{\varepsilon}$. $\Rightarrow$ $\forall n \geq N$ $\quad n^{-1} < \varepsilon$.

$n^{-1} \in (-\varepsilon, \varepsilon)$

$n^{-1} \in \mathcal{U}$

the only members of (n^{-1}) not in $\mathcal{U}$ are finitely many $\{1, \ldots, \frac{1}{N-1}\}$

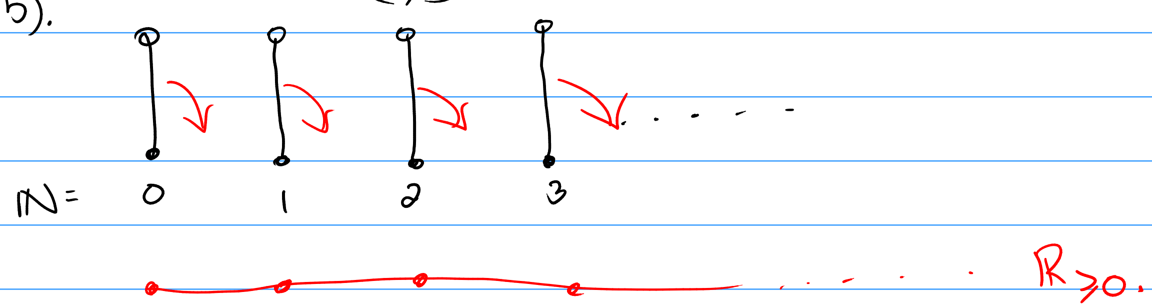
Take any $\mathcal{U} \ni 0'$. open then $\tilde{\mathcal{U}} = (\mathcal{U} \setminus \{0'\}) \cup \{0\}$ is open.

$(-\varepsilon, \varepsilon) \subset \tilde{\mathcal{U}}$

$\forall n \geq N$ $\quad n^{-1} \in \tilde{\mathcal{U}}$

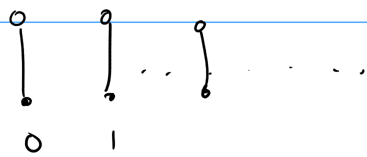
$n^{-1} \in \mathcal{U}$ since $n^{-1} \neq 0$.

b). $(n, x) \mapsto n + x$.



c)

uncountably many



Next week