Def 1.6 A topology is a subset $\tau \subseteq P(X)$ with 3 properties

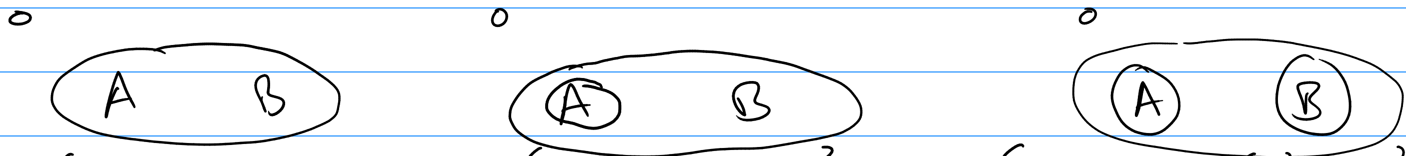
(iii) $\phi \in \tau$, $X \in \tau$

(ii) If $\{U_i\} \subseteq \tau$ then $\bigcup U_i \in \tau$

(i) If $\{U_i\}_{i=1,\dots,n} \subset \tau$ then $\bigcap U_i \in \tau$.

A set that "belongs to the topology" is called "open"

Most basic example. $X = \{A, B\}$



	$\tau = \{\phi, X\}$	$\tau = \{\emptyset, \{A\}, X\}$	$\tau = \{\phi, \{A\}, \{B\}, X\} = P(X)$
closed	$X, \emptyset$	$X, \{B\}, \emptyset$	$X, \{B\}, \{A\}, \emptyset$
	Trivial topology	Sierpiński space	Discrete topology
Connected:	✓	✓	X
metrisable:	X	X	$d(A,A) = d(B,B) = 0$, $d(A,B) = 1$
Hausdorff:	X	X	✓ $\{A\}$ is a n'hood of A, $\{B\}$ is a n'hood of B

Terminology.

Neighbourhood of x / Umgebung = an open set containing x

Cover / Überdeckung = a collection of open sets, whose union is X

$\{\{A\}, \{B\}\}$ is a cover of the discrete space

Closed / abgeschlossen A = $X \setminus A$ is an open set

Closure of V is the smallest closed set that contains V.

Trivial	$V = \{A\}$	$\overline{V} = X$
Discrete	$V = \{A\}$	$\overline{V} = \{A\}$
Sierpiński	$V = \{A\}$	$\overline{V} = X$

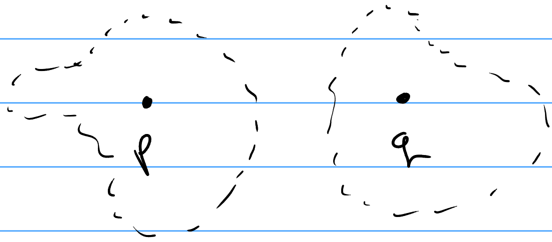
Closed sets of Sierpinski

X, $\{B\}$, $\emptyset$

weak / strong
" "
coarse / fine

If $\tau_1 \subseteq \tau_2$ then τ_1 is called weaker (coarser) than τ_2

Hausdorff



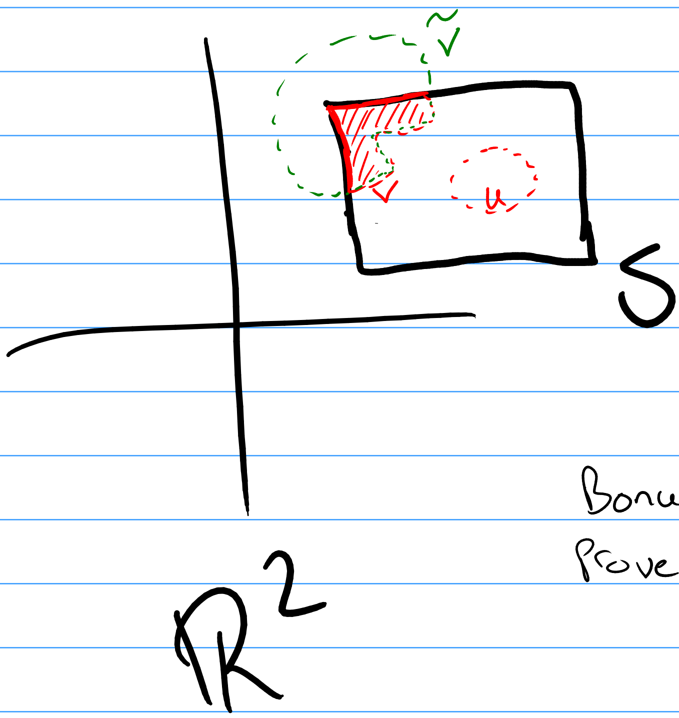
If every open n'hood of p and q intersect, X is not Hausdorff

Useful fact • every metric space is Hausdorff.
• every subspace of a metric space is a metric space.

Subspaces

(X, τ_X) is a topological space

If $Y \subset X$ $\quad \tau_Y := \{ U \mid U = Y \cap V \text{ and } V \in \tau_X \}$



U is an open set of S, because $U = U \cap S$ open in $\mathbb{R}^2$

$\tilde{V}$ is open set of $\mathbb{R}^2$

$V = \tilde{V} \cap S$ so V is open in S.

Bonus exercise for you to try later:
Prove τ_Y is a topology.

Let $I = [0,1) \subseteq \mathbb{R}$

What are the open and closed sets of I?

- I is open in I because $I = I \cap (-1, 1)$
- $(0.5, 1)$ is open in I because $(0.5, 1)$ is open in $\mathbb{R}$ and $(0.5,1) = I \cap (0.5, 1)$
- If U is an open set of $\mathbb{R}$ and $U \subset I$ then U is an open set of I. $U = I \cap U$

eg $U = (0.3, 0.5) \cup (0.7, 0.9)$

- $[0, 0.5)$ is open in I because $[0, 0.5) = I \cap \underbrace{(-1, 0.5)}_{\text{open in } \mathbb{R}}$
- $[0, a)$ and (a, b) are open in I, $0 \leq a \leq b \leq 1$.
- Claim: all open sets of I are unions of intervals of these forms.

Idea: Basis of topology

Often, the open sets of a space can be written as the union of a special type of open set.

eg if we want to prove "for every $x \in I$ there exist an neighbourhood of x ..."

$\Rightarrow$ instead prove for neighbourhoods of the form (a,b) or $[0,b)$.

Common example. $x \in \mathbb{R}^3$ Take any open neighbourhood U of x.

$\exists \varepsilon > 0$ so that $B(x, \varepsilon) \subset U$.

prove something about $B(x, \varepsilon)$