Partial Differential Equations

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Sheet 12

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40. Finite differences. Let $u \in W^{1,2}(B(0,1))$ be a weak solution of

$$L_0 u := \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u) = f \text{ in } B(0,1)$$

with $a_{ij} \in L^\infty(B(0,1))$ and $f \in L^2(B(0,1))$. Show that the finite difference

$$\partial_l^h u(x) := \frac{u(x+he_l) - u(x)}{h} \quad \text{for } x \in B(0,1-|h|)$$

is a weak solution of

$$L_0 \partial_l^h u(x) = \partial_l^h f(x) - \sum_{i,j=1}^{n} \partial_i(\partial_l^h a_{ij} \partial_j u(x+he_l)),\ x \in B(0,1-|h|).$$

41. An Interpolation inequality.

Let $(X, \|\cdot\|)$ be a Banach space that contains $C^1(\overline{B(0,2)})$. In other words, there exists a continuous, injective linear map $I : C^1(\overline{B(0,2)}) \hookrightarrow X$. Show that a constant $C(n) < \infty$ exists such that

$$\|u\|_{C^2(\overline{B(0,2)})} \leq C(n)\left(\|D^2 u\|_{L^\infty(\overline{B(0,2)})} + \|I(u)\|_X\right).$$

[Hint. Show that the embedding $C^2(\overline{B(0,2)}) \to C^1(\overline{B(0,2)}) \hookrightarrow X$ satisfies the assumptions of Ehrling's Lemma 3.3, ie that $T : C^2(B(0,2)) \to C^1(B(0,2))$ is compact and $\|u\|_{C^2(B(0,2))} = \|D^2u\|_{L^\infty(B(0,2))} + \|u\|_{C^1(B(0,2))}$.]

42. Equivalent Norms.

Let $U \subset C^2(B(0,2))$ be the subspace given by $U := \{u \in C^2(B(0,2)) \mid u(0) = 0, \nabla u(0) = 0\}$. Show that on U the norm $\|D^2u\|_{L^\infty(B(0,2))}$ is equivalent to the norm $\|u\|_{C^2(B(0,2))}$.

[Hint. $u(x) - u(0) = \int_0^1 \nabla u(t\cdot x)\cdot x \,\mathrm{d}t$.]

43. The interior Schauder Estimate.

At which place in the proof of the interior Schauder estimate 4.11 should we make a modification to instead prove the inequality

$$\|u\|_{C^{2,\alpha}(B(0,1))} \leq C(\Lambda, n, \alpha)\left(\|Lu\|_{C^{0,\alpha}(B(0,2))} + \|u\|_{L^1(B(0,2))}\right)$$

for $u \in C^{2,\alpha}(B(0,2))$? (Note in particular the last term on the right.)