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**35. A differerential equation with many weak solutions.**

Let $\Omega \subset \mathbb{R}^2$ be open and $L$ a (non-elliptic) differential operator on $\Omega$, defined by

$$Lu := \partial_1(\partial_2 u) - \partial_2(\partial_1 u).$$

Show that every $u \in W^{1,2}(\Omega)$ is a solution of $Lu = 0$ in the weak sense.

**36. Divergence and rotation.**

Let $\Omega \subset \mathbb{R}^n$ be open. We say that a vector field $f = (f_1, \ldots, f_n) : \Omega \to \mathbb{R}^n$ with $f_k \in L^2(\Omega)$, $k \in \{1, \ldots, n\}$ is a weak solution the differential equation $\nabla \cdot f = 0$ when

$$\int_\Omega (f \cdot \nabla\phi)\, d\mu = 0 \quad \text{for all} \quad \phi \in W_0^{1,2}(\Omega).$$

Now set $n = 3$. The *curl* (also called the *Rotation*) of a vector field $u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3$ with $u_k \in W^{1,2}(\Omega)$, $k \in \{1, 2, 3\}$ is defined to be

$$\nabla \times u := (\partial_2 u_3 - \partial_3 u_2,\, \partial_3 u_1 - \partial_1 u_3,\, \partial_1 u_2 - \partial_2 u_1),$$

in analogy to the cross product $\times$.

Show that the curl $f := \nabla \times u$ of $u$ is a weak solution of $\nabla \cdot f = 0$.

**37.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and

$$Lu = -\sum_{i,j=1}^n \partial_j(a_{ij}\partial_i u) + \sum_{i=1}^n c_i \partial_i u + du$$

be an elliptic operator with elliptic constant $\Lambda$ as per Def. 4.1. Define a bilinear form on $W_0^{1,2}(\Omega)$ by

$$a(u, v) := \int_\Omega \left( \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j v + \sum_{i=1}^n c_i(\partial_i u)v + duv \right) dx.$$

Show that there are constants $\beta > 0$ and $\gamma \geq 0$ that only depend on $\Lambda$ such that

$$\beta \|u\|^2_{W_0^{1,2}(\Omega)} \leq a(u, u) + \gamma \|u\|^2_{L^2(\Omega)}$$

for all $u \in W_0^{1,2}(\Omega)$.

[Hint: Young's inequality $xy \leq \frac{1}{2}(\frac{1}{\epsilon}x^2 + \epsilon y^2)$ for any $x, y \geq 0$ and $\epsilon > 0$ may be helpful.]

**38. On Friedrich's Theorem on the interior.**

Consider the real, open intervals $I := (-2, 2)$ and $J := (-1, 1)$. We choose a function $a \in L^\infty(I) \setminus W^{1,2}(J)$ wth $a \geq 1$, and let

$$u : I \to \mathbb{R}, \qquad u(t) := \int_0^t \frac{1}{a(x)} \, \mathrm{d}x.$$

**(a)** Show that $u \in W^{1,2}(I)$ and $u \notin W^{2,2}(J)$.

**(b)** Show that $u$ is a weak solution of $(au')' = 0$ on $I$.

**(c)** Why does this not contradict Friedrich's theorem on the interior?

**39. On the Cacciopoli inequality at the boundary.**

Complete the proof of the Cacciopoli inequality at the boundary (Theorem 4.7) from the lecture notes by adapting the proof of Theorem 4.5.