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Sheet 10

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33. On the weak solutions of elliptic differential equations.

Let $\Omega \subset \mathbb{R}^n$ be open, $u \in W^{1,2}(\Omega)$, $f \in W_0^{1,2}(\Omega)^*$, and L be an elliptic differential operator in the sense of Definition 4.1.

(a) State what it means for u to be a weak solution of $Lu \geq f$.

(b) Suppose that u is a weak solution of both

$$Lu \geq f \quad \text{and} \quad Lu \leq f, \tag{$*$}$$

in the sense of Definition 4.1. *Show* for all $v \in W_0^{1,2}(\Omega)$ that $\mathcal{L}(u, v) = -\langle f, v\rangle$. Note: there is requirement for v to be positive.

(c) *Show* that the following is a distribution:

$$C_0^\infty(\Omega) \ni \phi \mapsto -\mathcal{L}(u, \phi).$$

(d) *Show* that if $(*)$ holds then $Lu = f$ in the sense of distributions.

(e) Suppose now that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, $f \in L^2_{\mathrm{loc}}(\Omega)$ such that $\triangle u \geq f$ and $\triangle u \leq f$ hold in the weak sense. *Show* for all $\phi \in C_0^\infty(\Omega)$ that

$$\triangle(\phi u) = (\triangle\phi)u + 2\nabla\phi \cdot \nabla u + f\phi$$

holds in the sense of distributions.

34. Weak solutions of the Poisson equation.

In the following we demonstrate an example of functions $u, f \in C^0(\Omega)$ such that $\triangle u = f$ in the weak sense, but $u \notin C^2(\Omega)$. Let $B(0, \frac{1}{2}) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < \frac{1}{4}\}$ and $u(x, y) := (x^2 - y^2)\log|\log(r)|$ with $r = (x^2 + y^2)^{1/2}$.

(a) *Show* that $u \in C^2(B(0, \frac{1}{2})\backslash\{0\})$ and $\lim\limits_{r\to 0} u(x, y) = 0$. In other words, u extends to a continuous function on $B(0, \frac{1}{2})$.

(b) *Compute* the following derivatives of u on $B(0, \frac{1}{2})\backslash\{0\}$

$$\begin{aligned}
\frac{\partial}{\partial x}u(x, y) &= 2x\log|\log(r)| + (x^3 - y^2x)\frac{1}{r^2\log(r)}, \\
\frac{\partial^2}{\partial x^2}u(x, y) &= 2\log|\log(r)| + (5x^2 - y^2)\frac{1}{r^2\log(r)} - (x^4 - x^2y^2)\frac{2\log(r) + 1}{r^4(\log(r))^2}.
\end{aligned}$$

(c) *Argue* that $\frac{\partial^2}{\partial y^2}u(x, y) = -\frac{\partial^2}{\partial x^2}u(y, x)$ and hence

$$\triangle u = (x^2 - y^2)\left(\frac{4}{r^2\log(r)} - \frac{1}{r^2(\log(r))^2}\right).$$

Conclude therefore that $\lim\limits_{r\to 0}\triangle u(x, y) = 0$.

(d) Let $g \in C(B(0, \frac{1}{2}))$ be the continuous extension of $\triangle u$ on $B(0, \frac{1}{2})$. *Prove* that $\triangle u = g$ weakly on $B(0, \frac{1}{2})$. Thereby establish for any $\phi \in C_0^\infty(B(0, \frac{1}{2}))$ the formula

$$\int_{B(0,\frac{1}{2})\setminus B(0,\varepsilon)} u\triangle\phi \,\mathrm{d}\mu = \int_{B(0,\frac{1}{2})\setminus B(0,\varepsilon)} g\phi \,\mathrm{d}\mu + \int_{\partial B(0,\varepsilon)} (u\nabla\phi - \phi\nabla u)\cdot N \,\mathrm{d}\sigma.$$

(e) Take $\psi \in C_0^\infty(B(0, \frac{1}{2}))$ with $\psi(x, y) = 1$ in a neighbourhood of 0 and let

$$f := \psi g + 2\nabla\psi \cdot \nabla u + (\triangle\psi)u \in C(B(0, \tfrac{1}{2})).$$

Show with the help of Question 33(c) that $\triangle(\psi u) = f$ holds in the weak sense.