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30. Approximation by Sobolev functions.

(a) Let $\Omega = \{x \in \mathbb{R}^2 \mid |x| < 1\}$ and $\eta \in C_0^\infty(\Omega)$ be such that $\eta(x) = 1$ for $|x| \leq \frac{1}{2}$. Define $u(x) = \left(\log \frac{1}{|x|}\right)^{1/4} \cdot \eta(x)$. *Show* that $u \in W^{1,2}(\Omega)$.

(b) Let $\Omega \subset \mathbb{R}^n$ be open and $u, v \in W^{1,2}(\Omega)$.

(i) *Show* for $w(x) := \min\{u(x), v(x)\}$ that w also lies in $W^{1,2}(\Omega)$ and *determine* the weak derivatives of w.

(ii) *Show* using (i) that for $r(x) = \max\{u(x), v(x)\}$ that likewise $r \in W^{1,2}(\Omega)$.

(c) Let $\mathrm{id}_\Omega : \Omega \to \{0, 1\}$ be the indicator function of the open set $\Omega \subset \mathbb{R}^n$. *Prove* that when $u \in W^{1,2}(\Omega)$ then $w := \min\{u, \mathrm{id}_\Omega\} \in W^{1,2}(\Omega)$. *Calculate* the first derivatives of w.

(d) Show $L^\infty(\Omega) \cap W^{1,2}(\Omega)$ is dense in $W^{1,2}(\Omega)$.

[Hint. For $u \in W^{1,2}(\Omega)$ consider the sequence $u_n := \max\{-n \cdot \mathrm{id}_\Omega, \min\{u, n \cdot \mathrm{id}_\Omega\}\}$ and show $\lim\limits_{n\to\infty} \|u_n - u\|_{W^{1,2}(\Omega)} = 0$.]

31. Discontinuous functions in $W^{1,p}(\Omega)$.

Let $1 \leq p < n$ and $\Omega \subset \mathbb{R}^n$ be open. Give an example of a discontinuous function in $W^{1,p}(\Omega)$ (give a non-trivial example, not just a continuous function redefined on a set of measure zero).

32. Another approach to Sobolev inequalities.

Sobolev inequalities compare the "size" of ∇u with that of u. Therefore we want to express u in terms of its gradient.

(a) Let $u \in C_0^\infty(\mathbb{R}^n)$ and take polar coordinates $(r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ on $\mathbb{R}^n$. *Show*:

$$u(x) = -\frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} \int_0^\infty \partial_r u(x + r\theta)\, \mathrm{d}r\, \mathrm{d}\theta.$$

[Hint. First show $u(x) = -\int_0^\infty \partial_r u(x + r\theta)\, \mathrm{d}r$. A similar expression was computed in the proof of 3.25.]

(b) *Prove* further that

$$u(x) = \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{\langle x - y, \nabla u(y)\rangle}{|y - x|^n}\, \mathrm{d}y$$

and thereby establish the inequality $|u(x)| \leq \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\, \mathrm{d}y$.

[Hint. Set $y := x + r\theta$.]