Partial Differential Equations

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Sheet 7

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21. The dual space of $L^p(\mathbb{R}^n)$.

Let $1 \leq p \leq \infty$. The Banach space $L^p(\mathbb{R}^n)$ has the norm

$$\|\cdot\| : L^p(\mathbb{R}^n) \to \mathbb{R}, \quad f \mapsto \|f\|_p = \left(\int_{\mathbb{R}^n} |f|^p \mathrm{d}\mu\right)^{1/p}.$$

We will show that for q with $\frac{1}{p} + \frac{1}{q} = 1$ the map

$$j : L^q(\mathbb{R}^n) \to (L^p(\mathbb{R}^n))' = \mathcal{L}(L^p(\mathbb{R}^n), \mathbb{R}), \quad g \mapsto j(g) \text{ with } j(g)(f) = \int_{\mathbb{R}^n} fg \, \mathrm{d}\mu$$

is a linear isometry, i.e. $\|g\|_q = \|j(g)\|$ holds. One can then show that for $1 \leq p < \infty$ the dual space of $L^p(\mathbb{R}^n)$ is isometrically isomorphic to $L^q(\mathbb{R}^n)$.

(a) *Show*, with the help of the Hölders inequality that $j : L^q(\mathbb{R}^n) \to (L^p(\mathbb{R}^n))'$ is Lipschitz continuous with Lipschitz constant $L = 1$.

(b) Give a function f for any given function g such that $\|f\,g\|_1 = \|f\|_p \cdot \|g\|_q$.

(c) From the previous part, show also that $\|g\|_q \leq \|j(g)\|$ for all $g \in L^q(\mathbb{R}^n)\backslash\{0\}$ and therefore that j is an isometry.

22. Concerning the Lax-Milgram Theorem.

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain and $L : C_0^2(\Omega) \to C_0(\Omega)$ the elliptic operator

$$(Lu)(x) = -\mathrm{div}(A(x)\nabla u(x)) + c(x)u(x)$$

given in divergence form. Let $K > 0$ and $c(x) \geq K \;\; \forall x \in \Omega$. Show that L obeys the inequality

$$\langle Lu, u\rangle_{L^2(\Omega)} \geq C \cdot \|u\|^2_{W^{1,2}(\Omega)} \;\; \text{(for a constant } C > 0).$$

23. Sobolev spaces.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$x \mapsto \begin{cases} 1+x & \text{für } -1 \le x \le 0 \\ 1-x & \text{für } \quad 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Describe the first derivative of the distribution $F : C_0^\infty(\mathbb{R}) \to \mathbb{R}$, $\phi \mapsto F(\phi) = \int_{\mathbb{R}} f(x)\phi(x)\mathrm{d}x$.

(ii) Show that the second derivative of the distribution $F(\phi) = \int_{\mathbb{R}} f(x)\phi(x)\mathrm{d}x$ is a linear combination of Dirac distributions.

(iii) Show: $f \in W^{1,1}(\mathbb{R})$, but $f \notin W^{2,1}(\mathbb{R})$.

(b) Let $\Omega = \mathbb{R}^n$ and $u \in W^{2,1}_{\text{loc}}(\mathbb{R}^n)$ so that $\partial^\alpha u = 0$ for all α with $|\alpha| = 2$ in the weak sense. Show that u is affine, i.e. $u(x) = a \cdot x + b$ *a.e.* with $a \in \mathbb{R}^n$ und $b \in \mathbb{R}$.

[Hint. Denote by e_i the i-th unit vector. Show that $u_{e_i} \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$, by using $\nabla u_{e_i} = 0$ *a.e.* for all i and using Proposition 3.22.]

24. The Sobolev embedding theorem.

Show that $W^{1,1}((0,1)) \hookrightarrow C([0,1])$ is a continuous embedding.

[Hint. One shows that $\|u\|_\infty \le \|u\|_1 + \|u_1\|_1$ holds. Therefore define, for $(u, u_1) \in W^{1,1}((0,1))$, the function $U := \int_{x_0}^x u_1(t)\,\mathrm{d}t$ and prove: $U \in W^{1,1}((0,1)) \cap C([0,1])$ and $U - u \equiv \text{const}$. It then follows that $|u|$ obtains a minimum $x_0 \in [0,1]$. Finally, one can show $|u(x) - u(x_0)| \le \|u_1\|_1$ and estimate $\|u\|_\infty$ with the triangle inequality.]