Partial Differential Equations

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Sheet 6

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18. The space of Hölder-continuous functions, part 2.

Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$. These exercise show that one only needs to consider $C^{0,\alpha}(\Omega)$ for open sets Ω. Let $0 < \alpha \leq 1$ and $u \in C^{0,\alpha}(\Omega)$.

(a) Give the definition of uniform continuity (gleichmäßig stetig).

(b) Show that u is uniformly continuous.

(c) Show that there is a unique function $\widetilde{u} \in C^{0,\alpha}_{\text{loc}}(\overline{\Omega})$ with $\widetilde{u}|_\Omega = u$.

(d) Prove that

$$\sup_{x \neq y \in \overline{\Omega}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x-y|^\alpha}.$$

19. Compact Operators.

Let X, Y be Banach spaces. A linear, continuous mapping $T : X \to Y$ is called compact when for every bounded sequence $(x_m)_{m\in\mathbb{N}}$ in X there exists a subsequence $(x_{m_l})_{l\in\mathbb{N}}$ on which $(Tx_{m_l})_{l\in\mathbb{N}}$ converges.

(a) Show that a linear continuous mapping $T : X \to Y$ is compact exactly when the image of the unit ball $B(0,1) = \{x \in X \mid \|x\| < 1\}$ of X is relatively compact. (Recall that relatively compact means that the closure $\overline{T[B(0,1)]}$ is compact.)

(b) Let X be a Banach space and $\mathrm{id}_X : X \to X$ be the identity mapping. Show that id_X is a compact operator if and only if X is finite-dimensional.

20. A detail from the proof of Schauder's fixed point theorem.

Let K and $\tilde{K}$ be bounded, closed, convex subets of $\mathbb{R}^n$ with non-empty interiors. Prove that K and $\tilde{K}$ are homeomorphic.