Martin Schmidt
Ross Ogilvie

Partial Differential Equations
Sheet 5

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15. A detail from the proof of the weak maximum principle.

Let H be a real $n \times n$-matrix with

$$H = H^t \quad \text{und} \quad x^t H x \leq 0 \quad \forall x.$$

We will show that there is then a matrix D such that $H = -D \cdot D^t$.

(a) Show that the eigenvalues of any real, symmetric $n \times n$-matrix H must be real. Consider then the map

$$f : \mathbb{R}^n \to \mathbb{R}, \ x \mapsto x^t H x$$

restricted to the sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ and let f_{max} be the maximum value. Show for any $v \in \mathbb{S}^{n-1}$ with $f(v) = f_{max}$ that $Hv = f_{max} v$ holds.
(6 Points)

(b) Continuing the assumption that v is an eigenvector of H, show for the orthogonal complement $W := \langle v \rangle^\perp = \{x \in \mathbb{R}^n | \langle v, x \rangle = 0\}$ of v that $HW \subset W$ holds. Inductively prove then that there exists a matrix O and real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$H = O \cdot \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \cdot O^t.$$

(c) Show next that a matrix D exists such that $H = -D \cdot D^t$.

(d) Give the defintion for a second order differential operator to be elliptic.

(e) Define, for a constant matrix A, the second order differential operator L on $\mathbb{R}^n$.

$$(Lu)(x) := \operatorname{div}(A \nabla u(x))$$

Show that L is elliptic exactly when there is an invertible linear map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $L(u \circ \varphi) = (\triangle u) \circ \varphi$.

16. Second order differential operators

Let $a_{ij}, \tilde{a}_{ij}, b_i, \tilde{b}_i$ and $c, \tilde{c}$ be differentiable functions on the open set $\Omega \subset \mathbb{R}^n$. Any linear differential operator $L : C^2(\Omega) \to C(\Omega)$ of second order may be written as

$$(Lu)(x) = \sum_{i,j=1}^{n} a_{ij}(x) \partial_i \partial_j u(x) + \sum_{i=1}^{n} b_i(x) \partial_i u(x) + c(x) u(x). \tag{1}$$

This is called *general form* or *non-divergence form.* In contrast, we say that the operator is in *divergence form* when it is written as:

$$(Lu)(x) = \sum_{i=1}^{n} \partial_i \left(\sum_{j=1}^{n} \tilde{a}_{ij}(x) \partial_j u(x) + \tilde{b}_i(x) u(x) \right) + \tilde{c}(x) u(x).$$

(a) Show that every such differential operator may be written in divergence form. Give the relationship between the coefficient functions.

Now let $\widetilde{\Omega} \subset \mathbb{R}^n$ be another open set and $\varphi : \Omega \to \widetilde{\Omega}$ a C^2-diffeomorphism. That is, φ is bijective, and both φ and φ^{-1} are twice continuously differentiable.

(b) Show that $\tilde{L}(\widetilde{u}) \circ \varphi = L(\widetilde{u} \circ \varphi)$ defines a second order differential operator $\widetilde{L}$ on $\widetilde{\Omega}$ for $\widetilde{u} \in C^2(\widetilde{\Omega})$. You may do this by writing $\widetilde{L}$ is general form.
[Hint. We have that $(\widetilde{L}\widetilde{u})(\widetilde{x}) = \big(L(\widetilde{u} \circ \varphi)\big)(\varphi^{-1}(\widetilde{x}))$ so apply the chain rule.]

(c) Now suppose that Ω und $\widetilde{\Omega}$ are bounded and that both functions φ, φ^{-1} extended continuously to the closure $\overline{\Omega}$, $\overline{\widetilde{\Omega}}$ respectively. Under this hypothesis, show that $\widetilde{L}$ is an elliptic operator exactly when L is. (Note, the relationship between L and $\widetilde{L}$ is symmetric, so it suffices to prove one direction only.)

17. Hölder-continuous functions.

Let $\Omega \subset \mathbb{R}^n$ be open.

(a) Give the defintion for a function u to belong to the space of Hölder continuous functions $C^{0,\gamma}(\Omega)$.

(b) Suppose that $\gamma > 1$. Show that $u \in C^{0,\gamma}(\Omega)$ is differentiable and that $\nabla u \equiv 0$. This shows if Ω is connected and $\gamma > 1$ that $C^{0,\gamma}(\Omega)$ only contains the constant funtions.

(c) Suppose that u is continuously differentiable. Show that it is Hölder continuous for all $0 < \gamma \leq 1$.

(d) Choose some $0 < \alpha \leq 1$. Define $f : [0,1] \to \mathbb{R}$ by $x \mapsto x^\alpha$. To which Hölder spaces does f belong? Compute its norm.

(e) Now choose some $0 < \beta \leq 1$. Define $g : [0,\infty) \to \mathbb{R}$ by $x \mapsto x^\beta$. To which Hölder spaces does g belong? Compute its norm.

(f) Define $h : [0, 0.5] \to \mathbb{R}$ by $h(0) = 0$ and $h(x) = (\ln x)^{-1}$ otherwise. Show that this function is continuous but not Hölder continuous. Can you explain why?