Partial Differential Equations

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Sheet 4

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11. Harmonic functions on $B(0,1) \subset \mathbb{R}^2$.

By $B(0,1)$ we mean the open unit ball in $\mathbb{R}^2$ centred on 0.

(a) Let $u \in C^2(\overline{B(0,1)})$ be a $B(0,1)$ harmonic function, with the polar coordinate form $u = u(r,\varphi)$ (for $0 \le r \le 1$ and $0 < \varphi \le 2\pi$). Show that

$$\int_{\partial B(0,1)} \frac{\partial u}{\partial r}(x) \mathrm{d}\sigma(x) = 0$$

holds. *(8 Points)*

[Hint: As is familiar by now, begin with $\int_{B(0,1)} \triangle u d^2 x$ and apply the divergence theorem.]

(b) Guess a solution $u \in C^2(\overline{B(0,1)})$ for each of the following *Neumann Problems* or prove that there can be no solution.

(i) $\triangle u = 0$ on $B(0,1)$ with $\frac{\partial u}{\partial r} = \sin(\varphi)$ on $\partial B(0,1)$. *(6 Points)*

(ii) $\triangle u = 0$ on $B(0,1)$ with $\frac{\partial u}{\partial r} = \sin^2(\varphi)$ on $\partial B(0,1)$. *(6 Points)*

[Note: One may freely use the Laplace-Operator for a function $u = u(r,\varphi)$ in polar coordinates: $\triangle u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}$.]

12. Harmonic functions on $\Omega \subseteq \mathbb{R}^n$.

Let $\Omega \subseteq \mathbb{R}^n$ be a (open and connected) domain and $u : \Omega \to \mathbb{R}$ harmonic, i.e. $\Delta u = 0$ in Ω. Further, let $u(x) \le 1$ for all $x \in \Omega$.

Show that: in the case that there is an $x_0 \in \Omega$ with $u(x_0) < 1$, then it holds that $u(x) < 1$ for *all* $x \in \Omega$. *(8 Points)*

13. Harmonic functions spezieller Gestalt.

Let there be a twice-differentiable function $v : \mathbb{R}^+ \to \mathbb{R}$ as well as a fixed vector $y \in \mathbb{R}^n \setminus \{0\}$. We define the following two functions

$$u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R},\, x \mapsto v(\|x\|)$$
$$\text{und} \quad w : \mathbb{R}^n \setminus \{\tfrac{y}{\|y\|^2}\} \to \mathbb{R},\, x \mapsto v(\sqrt{1 + \|x\|^2\|y\|^2 - 2x \cdot y})\ .$$

Show that u is harmonic exactly when w is. *(10 Points)*

[Hint: Exercise 8(a)]

Please turn over.

14. A detail in the proof of the Poisson Representation Formula (Poissonschen Darstellungsformel).

We denote by $K(x,y)$ the Poisson kernel as in Abschnitt 2.3 of the lecture notes. This has the following properties (do *not* prove these properties again, refer to the lecture notes):

(i) $K(x,y) > 0$ for $x \in B(0,1),\ y \in \partial B(0,1)$.

(ii) $\int_{\partial B(0,1)} K(x,y)\mathrm{d}\sigma(y) = 1$ for $x \in B(0,1)$.

(iii) For all $y_0 \in \partial B(0,1)$, the map $y \mapsto K(x,y)$ converges in the limit $x \to y_0$, $x \in B(0,1)$ on compact subsets of $\partial B(0,1) \setminus \{y_0\}$ uniformly with repect to y.

Let a continuous function $u \in C(\partial B(0,1))$ be given. We define

$$\tilde{u} : B(0,1) \to \mathbb{R},\ x \mapsto \int_{\partial B(0,1)} K(x,y)u(y)\mathrm{d}\sigma(y)\ . \tag{$*$}$$

Show that the function $\tilde{u}$ can be extended continuously to the boundary $\partial B(0,1)$ and that the extension on $\partial B(0,1)$ agrees with u. *(12 Points)*

[Hint: For any given $x_0 \in \partial B(0,1)$ consider $x \in B(0,1)$ in a neighbourhood of x_0 and break the intgral $(*)$ into a piece close to x_0 and the "rest". Used the properties of K given above to show that the "rest" is small. For the part close to x_0 use the continuouity of u to approximate the function values of $u(y)$ and $u(x_0)$.]