Partial Differential Equations

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Sheet 3

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7. A detail from the proof of the mean value property of harmonic functions. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, $x_0 \in \mathbb{R}^n$, and $\partial B(x_0, r) := \{\, x \in \mathbb{R}^n \mid \|x - x_0\| = r \,\}$ for $r > 0$. The consider the function

$$\Phi(r) := \frac{1}{\sigma(\partial B(x_0, r))} \int_{\partial B(x_0,r)} f(x)\, \mathrm{d}\sigma(x)\ .$$

Show $\lim_{r\to 0} \Phi(r) = f(x_0)$. *(5 Points)*

8. Fundamental solution of the Laplace equation.

Let $n \geq 2$.

(a) Let $u \in C^2(\mathbb{R}^n)$ be rotationally symmetric. This means that $u(x) = v(\|x\|)$ for some twice continuously differentiable function $v : [0, \infty) \to \mathbb{R}$. Show then that

$$\triangle u(x) = \|x\|^{1-n} \cdot \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=\|x\|} (r^{n-1} \cdot v'(r)).$$

(4 Points)

(b) The volume of the unit ball in $\mathbb{R}^n$ is givn by $\omega_n := \int_{\{\, x\in\mathbb{R}^n \,|\, \|x\|\leq 1 \,\}} 1\, \mathrm{d}^n x$. Show that the function

$$\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R},\ x \mapsto \begin{cases} -\frac{1}{2\pi} \log(\|x\|) & \text{für } n = 2 \\ \frac{1}{n\,(n-2)\,\omega_n} \|x\|^{2-n} & \text{für } n \geq 3 \end{cases}$$

is harmonic. Φ is called the *Fundamental solution of the Laplace equation.* Show also:

$$\nabla \Phi = -\frac{1}{n\,\omega_n} \frac{x}{\|x\|^n}\ .$$

(6 Points)

9. Solution of the Poisson equation. Let $n \geq 2$ and Φ the fundamental solution of the Laplace equation as in Question 8. Further, let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function with compact support. We will show that the convolution of f with Φ is a solution of the Poisson equation $-\triangle u = f$ in $\mathbb{R}^n$.

(a) Establish that the integral $\int_{\mathbb{R}^n} f(x - y)\, \Phi(y)\, \mathrm{d}^n y$ is well-defined for every $x \in \mathbb{R}^n$ (even though Φ is not defined at the origin $0 \in \mathbb{R}^n$). Then show that the function

$$u : \mathbb{R}^n \to \mathbb{R},\ x \mapsto (f * \Phi)(x) := \int_{\mathbb{R}^n} f(x - y)\, \Phi(y)\, \mathrm{d}^n y$$

is twice continuously differentiable and that

$$\triangle u = \int_{\mathbb{R}^n} \triangle f(x - y) \cdot \Phi(y)\, \mathrm{d}^n y \qquad (*)$$

holds. *(4 Points)*

(b) Fix now an $\varepsilon > 0$. We now sepearate the integral $(*)$ into one part that contains the singularity from Φ and another part that is singularity free:

$$I_\varepsilon := \int_{B(0,\varepsilon)} \triangle f(x-y) \cdot \Phi(y)\, \mathrm{d}^n y \, ,$$
$$J_\varepsilon := \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \triangle f(x-y) \cdot \Phi(y)\, \mathrm{d}^n y \, .$$

Show that $\lim_{\varepsilon \to 0+} I_\varepsilon = 0$. *(6 Points)*

(c) For every J_ε, prove that the following estimate holds:

$$J_\varepsilon = -\int_{\partial B(0,\varepsilon)} f(x-y) \cdot \nabla_y \Phi(y) \cdot N \mathrm{d}\sigma(y) + L_\varepsilon \, ,$$

where L_ε is some expression that converges to zero as $\varepsilon \to 0$. *(6 Points)*

[Hint. Use Question 2(b), then divergence theorem, then again Question 2(b). Also use (from Question 8(b)) that Φ is harmonic on $\mathbb{R}^n \setminus B(0,\varepsilon)$.]

(d) Finally, establish that $\int_{\partial B(0,\varepsilon)} f(x-y) \cdot \nabla_y \Phi(y) \cdot N \mathrm{d}\sigma(y)$ is the mean value of f on $\partial B(x,\varepsilon)$, and therefore $-\triangle u = f$. *(4 Points)*

[Hint. Recall the formula for $\nabla\Phi$ from Question 8(b). The inward pointing unit normal field on $\partial B(0,\varepsilon)$ is given by $N(x) = -\frac{x}{\|x\|}$ and the volume σ_n of the unit sphere $\{\, x \in \mathbb{R}^n \mid \|x\| = 1 \,\}$ is $\sigma_n = n \cdot \omega_n$. Combine these facts with Question 7 to reach the conclusion.]

10. Subharmonic Functions.

Let $\Omega \subset \mathbb{R}^n$ be an open connected domain. A twice continuously differentiable function $v : \overline{\Omega} \to \mathbb{R}$ is callled *subharmonic*, when $-\triangle v \leq 0$ on Ω.

(a) Let $v : \overline{\Omega} \to \mathbb{R}^n$ be subharmonic. Show for all $x \in \Omega$ and $r > 0$ with $B(x,r) \subset \Omega$ that the following holds:

$$v(x) \leq \frac{1}{r^{n-1}\, n\, \omega_n} \int_{\partial B(x,r)} u(y)\, d\sigma(y) \, .$$

[Hint: Adapt the proof of the mean value property] *(5 Points)*

(b) Following from (a): Assume that v has a maximum on Ω. That is, there is a point $x_0 \in \Omega$ such that $v(x_0) = \sup_{x \in \Omega} v(x)$. Prove that v constant. *(4 Points)*

(c) Now suppose that $u : \overline{\Omega} \to \mathbb{R}$ is a harmonic function. Show:

(i) $\|\nabla u\|^2$ is subharmonic. *(3 Points)*

(ii) if $f : \mathbb{R} \to \mathbb{R}$ is a smooth, convex function then $f \circ u$ is subharmonic. *(3 Points)*