Partial Differential Equations

Martin Schmidt
Ross Ogilvie

Sheet 2

20.2.2020

4. Derivative of the Inverse. Let $a < b$ and $A : (a,b) \to \mathcal{L}(\mathbb{R}^n)$ be a differentiable map from the real interval (a,b) to the space of continuous linear maps $\mathbb{R}^n \to \mathbb{R}^n$.

(a) Define what it means for this map to be differentiable. *(1 Point)*

(b) For every $t \in (a,b)$, suppose that $A(t) \in \mathcal{L}(\mathbb{R}^n)$ is invertible. Show then that the map

$$A^{-1} : (a,b) \to \mathcal{L}(\mathbb{R}^n),\ t \mapsto (A(t))^{-1}$$

has the derivative given by

$$\frac{\mathrm{d}}{\mathrm{d}t} A^{-1}(t) = -A^{-1}(t) \cdot \frac{\mathrm{d}}{\mathrm{d}t} A(t) \cdot A^{-1}(t).$$

In particular, A^{-1} is differentiable. *(5 Points)*

5. On Distributions.

(a) Show that

$$F : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto \int_{\mathbb{R}} x^3 \cdot \phi''(x)\mathrm{d}x$$

is a distribution on $\mathbb{R}$, and define a function $f : \mathbb{R} \to \mathbb{R}$ with

$$F(\phi) = \int_{\mathbb{R}} f(x) \cdot \phi(x)\mathrm{d}x \text{ for all } \phi \in C_0^\infty(\mathbb{R}).$$

(4 Points)

(b) Show that the Dirac-Distribution

$$\delta : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto \phi(0)$$

is indeed a distribution on $\mathbb{R}$ and prove that there does *not* exist a function $g : \mathbb{R} \to \mathbb{R}$ with

$$\delta(\phi) = \int_{\mathbb{R}} g(x) \cdot \phi(x)\mathrm{d}x \text{ for all } \phi \in C_0^\infty(\mathbb{R}).$$

(2+4 Points)

(c) Calculate the derivatives F' and δ' of the distributions in parts (a) und (b) respectively. *(2+2 Points)*

6. On Convolutions.

(a) Let $f(x) = 1$ for $-1 \leq x \leq 1$ and 0 otherwise. Compute $f * f$. *(2 Points)*

(b) Show that the convolution of C_0^∞-functions on $\mathbb{R}^n$ is a bilinear, commutative, and associative operation. *(2+3+4 Points)*

(c) Denote a constant function on $\mathbb{R}$ by 1, the derivative of the Dirac distribution by δ' (see 5(c)), and the Heaviside function $H : \mathbb{R} \to \mathbb{R}$. This is defined as $H(x) := 1$ for $x \geq 0$ and $H(x) := 0$ for $x < 0$. Show that $1 * (\delta' * H) \neq (1 * \delta') * H$. This shows that the convolution of distributions with non-compact support (on $\mathbb{R}$) is not necessarily associative, even when it is well-defined.

(7 Points)