Partial Differential Equations

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Sheet 1

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1. The linear transport equation

Let $b \in \mathbb{R}^n$. The (homogeneous) linear transport equation with direction b is given by the following partial differential equation of first order:

$$\dot{u} + b \cdot \nabla u = 0 \, . \qquad (*)$$

This is a differential equation of $u = u(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, where $\dot{u}$ denotes the derivative of u with respect to $t \in \mathbb{R}$ and the gradient ∇u is taken with respect to $x \in \mathbb{R}^n$.

(a) Prove: If $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a solution of $(*)$, then u is constant on each of the parallel lines with direction $(b, 1) \in \mathbb{R}^n \times \mathbb{R}$. *(4 points)*

(b) Let $g \in C^1(\mathbb{R}^n)$. Prove that $u(x,t) := g(x - tb)$ is the *unique* solution of $(*)$ satisfying $u(\cdot, 0) = g$. *(5 points)*

2. Laplacian and Laplace equation

(a) Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $f \in C^2(\Omega)$ and $g \in C^1(\Omega)$. Show that

$$g \triangle f = \nabla \cdot (g \nabla f) - \nabla f \cdot \nabla g$$

holds. *(4 points)*

(b) Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain. Let $u \in C^2(\overline{\Omega})$ be a solution of the *boundary value problem*

$$\triangle u = 0 \quad \text{with} \quad u|_{\partial\Omega} = 0 \, .$$

Show $u \equiv 0$. *(5 points)*

[Hint: Investigate $\int_\Omega u(\triangle u) \mathrm{d}\mu$ with the help of task 2(a) und the Gauss' divergence theorem.]

Turn the page, please.

3. Extremals of the Dirichlet integral

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Let $u \in C^2(\overline{\Omega})$ minimize the Dirichlet integral $\int_\Omega \|\nabla u\|^2$ among $C^1(\overline{\Omega})$ functions with fixed boundary values, i.e. for each function $v \in C^1(\overline{\Omega})$ with $v|_{\partial\Omega} = u|_{\partial\Omega}$, we have

$$\int_\Omega \|\nabla u\|^2 \leq \int_\Omega \|\nabla v\|^2 .$$

The aim of this exercise is to show that u is harmonic. We proceed as follows:

Take an arbitrary function $\lambda \in C^1(\overline{\Omega})$ whose support $\mathrm{supp}(\lambda) := \overline{\{ x \in \overline{\Omega} \mid \lambda(x) \neq 0 \}}$ is contained in Ω. We consider the family of functions $(u_t)_{t\in\mathbb{R}}$ with

$$u_t : \overline{\Omega} \to \mathbb{R},\ x \mapsto u(x) + t\lambda(x)$$

as well as the differentiable function

$$f : \mathbb{R} \to \mathbb{R},\ t \mapsto \int_\Omega \|\nabla u_t\|^2 .$$

Now prove the following steps.

(a) Show $\frac{\partial}{\partial t}\big|_{t=0} (\|\nabla u_t\|^2) = 2\nabla u \cdot \nabla\lambda$. *(4 points)*

(b) Show with the help of (a) that $f'(0) = -2\int_\Omega \lambda \triangle u$. *(4 points)*

[Hint: Task 2(a) and the Gauss' divergence theorem.]

(c) On the other hand, deduce from the minimal property of u that $f'(0) = 0$. *(2 points)*

(d) Combine (b) and (c) to deduce that $\triangle u|_\Omega = 0$. Hence, u is harmonic.

Here, you may use without proof that for each $x \in \Omega$ and for each open neighbourhood $U \subset \Omega$ of x, there is a function $\lambda \in C^1(\overline{\Omega})$ satisfying $\lambda \geq 0$, $\mathrm{supp}(\lambda) \subset U$ und $\lambda(x) = 1$. Apply (b) and (c) to such a λ in order to prove that $\triangle u(x) = 0$. *(4 points)*

Note that these exercise sheets will become available on Thursdays on the website of the chair "Lehrstuhl Mathe III"

`http://wim.uni-mannheim.de/schmidt`→Lehre→Aktuelles Semester → Partial Differential Equations.

Your answers should be submitted on the following Thursday at the lecture.

The points for each question are to give you an indication of the amount of work that should be required.

For any further questions concerning the exercises and tutorials, you can contact Ross Ogilvie. E-Mail: r.ogilvie@math.uni-mannheim.de).