Funktionentheorie II – Exercise Set 9

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As requested, I have included extra questions at the end. Please make sure you do the regular questions to keep up with the lectures and these will be the one we will discuss. Feel free to email me questions or arrange a time to talk if stalks are still confusing you.

Question 9.1 (*). This question is to recall the idea of an exact sequence from algebra. This is an effective method of organising information about kernels and images. Feel free to skip this question if you are already comfortable with this style. Let A, B, and C be abelian groups. Suppose that there exist homomorphisms ϕ and ψ such that

$$0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$$

is an exact sequence. State what properties ϕ and ψ must have for this sequence to be exact. What is the relationship between the groups A, B, and C?

Question 9.2. In this question we explore morphisms of sheaves and how some constructions for abelian groups carry over. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a topological space X. Define $\ker \varphi$ by

$$(\ker \varphi)(U) := \{f \in \mathscr{F}(U) \mid f \in \ker \varphi_U\}.$$

and $\operatorname{img}^P \varphi$ by

$$(\operatorname{img}^P \varphi)(U) := \{g \in \mathscr{G}(U) \mid g \in \operatorname{img} \varphi_U\}.$$

a. Prove that both $\ker \varphi$ and $\operatorname{img}^P \varphi$ are presheaves on X.

b. Show further that $\ker \varphi$ is always sheaf.

c. (*) However, use the example of $X = \mathbb{C}$, $\varphi : \mathscr{O} \to \mathscr{O}^*$ where $f(z) \mapsto \exp(2\pi i f(z))$, $U = \mathbb{C}^\times$, and $f(z) = z \in \mathscr{O}^*(U)$ to show that $\operatorname{img}^P \varphi$ is not a sheaf. (compare Beispiel 3.12)

Because of the previous exercise, Definition 3.11 in the lecture notes defines the image sheaf $\operatorname{img} \varphi$ to be the smallest sheaf that contains $\operatorname{img}^P \varphi$.

We say that φ is injective if $\ker \varphi = 0$ and that it is surjective if $\operatorname{img} \varphi = \mathscr{G}$.

d. Prove that φ is injective if and only if $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open sets $U \subset X$.

e. Show how the morphism φ induces a homomorphism of groups $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$.

f. Prove that φ is injective (resp. surjective) if and only if φ_x is injective (resp. surjective).

g. Show that $\exp : \mathscr{O} \to \mathscr{O}^*$ is surjective.

Let us organise the facts we have just shown into a statement about exact sequences of sheaves. Suppose that $0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \to 0$ is a short exact sequence of sheaves.

h. Show for every open set $U \subset X$ that the sequence

$$0 \to \mathscr{F}(U) \xrightarrow{\varphi} \mathscr{G}(U) \xrightarrow{\psi} \mathscr{H}(U)$$

is exact.

i. Illustrate by way of an example that $\mathscr{G}(U) \xrightarrow{\psi} \mathscr{H}(U)$ need not be surjective. (Hint. use again the example from part c. Or come up with another example if you like.)

Question 9.3. In this question we examine the definitions related to cochains. Choose your favourite Riemann surface X and take a open cover U of it. To be interesting, don't take the cover to be simply $\mathfrak{U} = \{X\}$, though it could be interesting to see what happens when X is a member of the cover. For example, your favourite Riemann surface might be the annulus A_2 and the cover might be $A_2 \setminus \mathbb{R}^+ \cup A_2 \setminus \mathbb{R}^-$

a. Give an example of a 0-cochain and a 1-cochain.

b. Compute the coboundary operators δ^0 and δ^1 applied to your examples respectively.

c. Were your examples coboundaries and/or cocycles? Why or why not?

d. On any Riemann surface, show that every 1-boundary is a 1-cocycle.

More germ and stalk questions

I would actually do these next two questions in the order: 9.4 a b, 9.5 a b c d, 9.4 c d e, 9.5 e f g, 9.4 f

Question 9.4 (*). Here is a step-by-step question about germs and stalks that expands Question 8.9 (Aufgabe 3.10). Take a topological space X and a presheaf $\mathscr{F}$. Consider the space of "pairs"

$$(U, f) \in \coprod_{U \in \tau} \mathscr{F}(U) =: \mathrm{Pair}(X),$$

where τ is the set of open sets of X (its topology) and this is disjoint union. Strictly speaking the U is unnecessary information because a section comes from a particular open set U, but it is useful to group these together.

For every point $p \in X$ we define an equivalence relation $\sim_p$ on $\mathrm{Pair}(X)$ by

$$(U, f) \sim_p (V, g) \Leftrightarrow \exists W \subset U \cap V, p \in W \text{ such that } f|_W = g|_W.$$

The equivalence class $[(U, f)]_p$ is called the germ of f at p and the set of equivalence classes $\mathscr{F}_p := \mathrm{Pair}(X)/\sim_p$ is the stalk of $\mathscr{F}$ at p.

a. Show that $\sim_p$ is an equivalence relation. That is, show it is symmetric, reflexive (both rather trivial), and transitive (a bit harder).

b. Show that if $p \in V \subset U$ then $(U, f) \sim_p (V, f|_V)$. This is a useful trick that simplifies many proofs involving germs.

c. Suppose that $[(U, f)]_p = [(U', f')]_p$ and $[(V, g)]_p = [(V', g')]_p$ are two germs with two representatives. Prove that

$$(U \cap V, (f|_{U\cap V}) + (g|_{U\cap V})) \sim_p (U' \cap V', (f'|_{U'\cap V'}) + (g'|_{U'\cap V'})).$$

Hint. Simplify the proof by first finding a sets U'' and V'' such that $f|_{U''} = f'|_{U''}$ and $g|_{V''} = g'|_{V''}$. Then use the fact that restriction is a homomorphism and the previous part.

d. Finish defining a group structure on the stalk $\mathscr{F}_p$.

e. Choose any open set U and a point $p \in U$. Show that projection π_p from $\mathscr{F}(U)$ to $\mathscr{F}_p$ given by $(U, f) \mapsto [(U, f)]$ is a group homomorphism.

f. Suppose that $f, g \in \mathscr{F}(U)$ are two sections of a sheaf. Show that $f = g$ if and only if $\pi_p(f) = \pi_p(g)$ for all points $p \in U$.

   Hint. Every point p has a neighbourhood W_p where f and g restrict to become equal. The collection $\{U \cap W_p \mid p \in U\}$ is a cover of U. It is essential that $\mathscr{F}$ is a sheaf.

Question 9.5 (*). Now we give a example that is slightly more difficult that the stalk of constant functions. It should show you that a stalk contains more information that just a function value. Let $X = \mathbb{R}$ and define the presheaf of linear functions

$$\mathscr{L}(U) = \{f : U \to \mathbb{R},\ f(x) = ax + b \mid a, b \in \mathbb{R}\}.$$

a. Choose $p = 0$ and the following section of $U = (-1, 1) \cup (2, 3)$:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \in (-1, 1) \\ 3 & \text{if } x \in (2, 3) \end{cases}.$$

   Show that $(U, f) \sim_p (\mathbb{R}, 2x + 1)$

b. Again let $p = 0$ but now take any open set U containing p. Prove that any section $f \in \mathscr{F}(U)$ is equivalent to $(\mathbb{R}, ax + b)$ for some constants $a, b \in \mathbb{R}$.

c. Prove that $(\mathbb{R}, ax + b) \sim_p (\mathbb{R}, cx + d)$ if and only if $a = c$ and $b = d$.

d. Hence show that the stalk $\mathscr{L}_p$ is isomorphic to the additive group $\mathbb{R}^2$.

e. Do the above questions depend on the choice of point p? Hence what is the stalk of $\mathscr{L}$ at any point $p \in \mathbb{R}$.

f. Let $\mathscr{P}_n$ be presheaf of polynomials of degree at most n. Find a representative for each germ and thereby calculate its stalks.

Question 9.6 (*). This questions expands on the explanation in the lectures that $\mathscr{O}_p \cong \mathbb{C}\{z\}$. Let X be a Riemann surface and choose any point $p \in X$. Let $\phi : U \to \mathbb{C}$ be a coordinate neighbourhood such that $\phi(p) = 0$. Further suppose that U is simply connected (this is not necessary, but avoids dealing with certain issues).

a. Show that any pair (V, f), with $p \in V$, is equivalent (in the sense of germs at p) to $(W, g \circ \phi)$ for some open set $W \subset U$ and holomorphic function $g : \phi(W) \to \mathbb{C}$.

b. Let $A \subset \mathbb{C}$ and $q \in A$. Prove that every holomorphic function on A is equivalent to its Taylor series at q.

c. Show that two convergent power series at q are equivalent if and only if they are equal.

d. Hence conclude that $\mathscr{O}_p \cong \mathbb{C}\{z\}$.