Funktionentheorie II – Exercise Set 8

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Questions marked with * are optional.

Throughout these questions G is an abelian group. In every question where the family of groups is a set of functions, the restriction homomorphisms ρ_V^U are taken to be the restriction of functions $f \mapsto f|_V$.

The Definition of Sheaves and Examples:

Question 8.1. Here is the classic question first question every student of sheaf theory must think about. Let X be a topological space with two connected components X_1 and X_2. We have already seen that the family of groups $\mathscr{G}$ defined by $G(U) := \{f : U \to G \,|\, f \text{ is locally constant}\}$ for any open set $U \subset X$ is a sheaf.

Define the family of groups $\tilde{G}$ by $\tilde{G}(U) := \{f : U \to G \,|\, f \text{ is constant}\}$ for any open set $U \subset X$. Show that this is a presheaf. Either prove that it is a sheaf or give a counterexample.

Question 8.2. It becomes very tedious to check that whether or not something is a (pre-)sheaf or not. In this question we automate the proof for sheaves of functions. Suppose that X is a topological space and $\mathscr{F}$ is defined by $\mathscr{F}(U) := \{f : U \to G \mid f \text{ has property } P\}$ for any open set $U \subset X$. From the previous question, 'f is constant' and 'f is locally constant' are examples of properties.

a. Suppose that property P is *restrictable.* That means if $f : U \to G$ has property P and $V \subset U$ is open, then $f|_V$ also has the property P. Show that $\mathscr{F}$ is a presheaf.

b. Suppose further that property P is *local.* That means the following: Take any open set $U \subset X$ and a function $f : U \to G$. Then f has property P if P holds for all restrictions $f|_{U_i}$ for any open cover $\{U_i\}$ of U. Show that $\mathscr{F}$ is a sheaf.

c. Prove that $\mathscr{C}$, $\mathscr{E}$, $\mathscr{O}$, $\mathscr{M}$ are sheaves.

d. Explain why 'f is constant' is not a local property.

e. Is 'f is a bounded function' a local property? Hence, is the family

$\mathscr{B}(U) := \{f : U \to \mathbb{R} \mid f \text{ is bounded}\}$ a presheaf, sheaf, or neither? Prove this or give a counter-example.

f. Take the family of groups $\mathscr{L}$ on the space $X = [0,1]$ defined by

$$\mathscr{L}(U) = \left\{ f \in L^1(U) \,\middle|\, \int_U f = 0 \right\}.$$

Is this a presheaf, sheaf, or neither?

g. Consider the family $\mathscr{F}(X,G)$ defined as $\mathscr{F}(X,G)(U) := \{f : U \to G\}$, the set of all functions from U to G. Is this a presheaf, sheaf, or neither?

Question 8.3. Sometimes students think there is a connection between the sheaf axiom and analytic continuation. There isn't, as this question tries to show. Consider $X = \mathbb{C}^\times$ and the open cover

$$U_j := \left\{ re^{i\theta} \in \mathbb{C} \,\middle|\, 0 < r,\ \frac{j\pi}{2} < \theta < \frac{(j+2)\pi}{2} \right\}$$

for $j \in \mathbb{Z}$.

a. Consider the function $f(r) = \ln r$ defined for $r \in \mathbb{R}^+$. What is its unique holomorphic extension f_{-1} on U_{-1}. Continuing the function further, give a family of holomorphic functions $f_j \in \mathscr{O}(U_j)$ such that f_j and f_{j+1} are equal on $U_j \cap U_{j+1}$.

b. Why doesn't the sheaf axiom of $\mathscr{O}$ applied to $f_j \in \mathscr{O}(U_j)$ give the existence of a global holomorphic function $\ln z$ on $\mathbb{C}^\times$?

c. Consider $g(z) = z$ on U_0 and $h(z) = e^z$ on U_2. Apply to sheaf axiom to get a section of $\mathscr{O}(\mathbb{C} \setminus \mathbb{R})$.

Question 8.4 (*).

a. Supose that X is a compact Riemann surface. What are the global sections $\mathscr{O}(X)$? Let $U \subset X$ be an open set. Is ρ^X_U injective, surjective, both, or neither?

b. More generally, if X is any Riemann suface and U, V are two open sets, what can be said about the restriction homomorphisms of $\mathscr{O}$?

c. Now consider the sheaf of smooth functions. Are its restriction homomorphisms injective, surjective, both, or neither?

Question 8.5. There is a comment in the lecture notes that $\mathscr{M}$ is not a presheaf of fields. In fact sheaves of fields are very rare. Let U, V be two open and disjoint sets in a Riemann surface X. Find a pair of meromorphic functions $f, g \in \mathscr{M}(U \cup V)$ such that $fg = 0$. What does this say about the sheaves of functions to fields generally, for example $\mathbb{C}$?

Note that $\mathscr{M}(X)$ is a field however. What condition on U ensures that $\mathscr{M}(U)$ is a field?

Questions about Germs and Stalks:

Question 8.6. Consider again the sheaf of constant functions $\mathscr{G}$ and the presheaf of global constant functions $\tilde{G}$ on a topological space X, from Question 8.1.

a. Write out what it means for two sections of this sheaf to have the same germ at a point $p \in X$.

b. What is the stalk of $\mathscr{G}$ at $p \in X$?

c. Compute the stalk of $\tilde{G}$ at $p \in X$.

Question 8.7. When X is a Riemann surface we saw in lectures that the stalk of $\mathscr{O}$ at any point is isomorphic to the ring of convergent power series. Consider in contrast the sheaf of smooth functions $\mathscr{E}$ on the space $X = (-1, 1)$ and in particular the function

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ e^{-x^{-2}} & \text{for } x > 0. \end{cases}$$

Prove that the pair (X, f) is not equivalent to $(X, 0)$ in the sense of germs at 0. What does this tell you about the stalk of the sheaf of smooth functions?

Question 8.8. In this question we define a *skyscraper sheaf*. Let X be a topological space and $q \in X$ a chosen point. Define $i_q(G)$ by

$$i_q(G)(U) = \begin{cases} G & \text{if } q \in U \\ 0 & \text{otherwise.} \end{cases}$$

a. Calculate the stalk of $i_q(G)$ at q and at other points $p \neq q$. This should explain the 'skyscraper'.

b. Describe the skyscraper sheaf in terms of a family of groups of functions to G.

Question 8.9. Take a topological space X, a presheaf $\mathscr{F}$, an open set U, and a point $p \in U$. For any section $f \in \mathscr{F}(U)$ there is a natural projection from (U, f) to its equivalance class of germs at p. This gives a projection π_p from $\mathscr{F}(U)$ to $\mathscr{F}_p$.

a. (*) Define a group structure on the stalk $\mathscr{F}_p$ such that the projection π_p is a group homomorphism.

b. Suppose that $f, g \in \mathscr{F}(U)$ are two sections of a sheaf. Show that $f = g$ if and only if $\pi_p(f) = \pi_p(g)$ for all points $p \in U$.

Off-topic for this Course:

Question 8.10 (*). All the previous examples, and all Riemann surfaces, are well-behaved topologically. So in this course sheaves will be just a language to organise familiar spaces of functions. However in this question we give an example of a space that is a little strange. In algebraic geometry it is common to consider spaces that have points which are not closed, ie $\overline{\{x\}} \neq \{x\}$. The intuition is that the closed points are the 'actual' points of the space but the non-closed points are giving extra information about which closed points belong to the same algebraic component. Our example here attempts to mimic some of their behaviour. Let $X = \{a, b, C\}$ with open sets

$$\emptyset, \{C\}, \{a, C\}, \{b, C\}, X$$

and closed sets

$$X, \{a, b\}, \{b\}, \{a\}, \emptyset.$$

For example, the closure of $\{C\}$, the smallest closed set containing it, is X. To define a sheaf $\mathscr{F}$ on X we need to give four groups and the restriction homomorphisms between them:

$$
\begin{tikzcd}
 & G = \mathscr{F}(X) \arrow[dl, "\gamma_a"'] \arrow[dr, "\gamma_b"] \arrow[dd, "\rho"] & \\
H_a = \mathscr{F}(\{a, C\}) \arrow[dr, "\kappa_a"'] & & H_b = \mathscr{F}(\{b, C\}) \arrow[dl, "\kappa_b"] \\
 & K = \mathscr{F}(\{C\}) &
\end{tikzcd}
$$

a. What conditions on the groups and homomorphisms are needed to make $\mathscr{F}$ a sheaf? In particular, show G is a subgroup of $H_a \oplus H_b$ and use the map $\psi(h_a, h_b) = \kappa_a(h_a) - \kappa_b(h_b)$.

b. Suppose that we choose $K = 0$. What does this force G to be? How would you describe this sheaf in terms of functions on $\{a, b\}$? (Compare this to the sheaf on a disconnected space from a previous question)

c. Suppose now that $K = H_a = H_b$ and that $\kappa_a = \kappa_b = \mathrm{id}$. Show that the space of global sections is also K. Hence every local section extends to a unique global section.

d. Generalise from previous examples. Suppose that $H_a = A \oplus K$ and $H_b = B \oplus K$ with restriction given by projection to the K factor. Find the form of G.

e. Opine in what way K is acting as a form of global constraints that is not possible to match with a sheaf on the space $\{a, b\}$ with the discrete topology.