Funktionentheorie II – Exercise Set 7

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Question 7.1. In this question we will prove Schwarz's Lemma: Let $\mathbb{D}$ be the open unit disc and $f : \mathbb{D} \to \mathbb{C}$ such that $f(0) = 0$ and $|f(z)| \leq 1$. Then $|f(z)| \leq |z|$.

a. Consider the function defined by

$$g(z) = z^{-1} f(z) \text{ for } z \neq 0, \text{ and } g(0) = f'(z).$$

Show that g is a holomorphic function on the disc.

b. Apply the maximum modulus principle to g on the ball $B(0, r)$ and finish the proof of the lemma.

c. Further, suppose there is some point $z_0 \in \mathbb{D} \setminus \{0\}$ where $|f(z_0)| = |z_0|$. Prove that g is constant and hence $f(z) = az$.

Question 7.2. In this question we will use the previous lemma to find all the automorphisms of the unit disc. This is result is known as the Schwarz-Pick theorem. Assume that $f : \mathbb{D} \to \mathbb{D}$ is a biholomorphism and define $z_0 = f(0)$.

a. Check that the Möbius transformation

$$M(z) = \frac{z - z_0}{\overline{z_0} z - 1}$$

is a map from the unit disc to itself. Clearly $M(z_0) = 0$.

b. Let $\tilde{f} = M \circ f$. Apply the Schwarz lemma twice to conclude that $\left|\tilde{f}(z)\right| = |z|$. Hence prove that f must be a Möbius transformation. Hint. This is where we use that f is a biholomorphism.

c. By composition with the Cayley transform, argue that any biholomorphism of the upper half plane with itself is also a Möbius transformation. What conditions hold on the coefficients of this transformation?

Question 7.3. In the previous tutorial we characterised annuli as biholomorphic to the set $\{1 < |z| < R\}$ for exactly one $R > 1$. In this question we consider two similiar looking Riemann surfaces.

a. Consider the action $\mathbb{Z} \times \mathbb{C} \to \mathbb{C} : (n, z) \mapsto z + n$. This defines the cylinder $\mathbb{C}/\mathbb{Z}$, a Riemann surface. Search the lecture notes to find a mapping between $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}^\times$, or produce such a map yourself. Call this map ϕ.

b. This action also restricts to give a well defined action on the upper half plane $\mathbb{H}$. Does this fit the form you described in the previous question? Which theorem tells us that $\mathbb{H}/\mathbb{Z}$ is a Riemann surface?

c. What is the image $\phi(\mathbb{H})$?

d. Use these two examples to improve Theorem 2.30 (the annulus theorem).