Funktionentheorie II – Exercise Set 6

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Question 6.1. We have seen previously that the real part of a holomorphic function is harmonic. In this question we revisit the proof of Theorem 2.1. Let X be a Riemann surface and $u : X \to \mathbb{R}$ a twice continuously differentiable function.

a. Show that there can be at most one holomorphic function whose real part is u.

b. Choose a point $z_0 \in X$ and an open and simply connected neighborhood Ω of z_0. Look through the proof of Theorem 2.1. Write a simple formula for the holomorphic function f on Ω such that $\operatorname{Re} f = u$, in terms of the function g defined therein.

d. Why is the assumption that Ω is simply connected necessary? Give a criterion for the existence of the holomorphic function f to be well-defined on all of X.

e. What is the connection to Question 5.4 and the Hodge star operator?

Question 6.2. In this question we classify (well-behaved) maps between annuli. An annulus is a set $\{z \in \mathbb{C} | R_0 < |z - a| < R_1\}$ for positive real numbers $R_0 < R_1$ and center $a \in \mathbb{C}$.

a. Show every annulus is biholomorphic to an annulus with $R_0 = 1$ centered on the origin.

Hence we define $A_R := \{z \in \mathbb{C} | 1 < |z| < R\}$, with $R > 1$. Suppose that $f : A_R \to A_S$ is a surjective holomorphic function.

b. Give a biholomorphic map from A_R to itself that exchanges the two boundary circles.

c. Prove that if a sequence of points $z_k \in A_R$ tends to a point in the boundary of A_R then the limit points of $f(z_k)$ lie in the boundary of A_S.

The next step we would like to take would be to extend f to a continuous function between the closed annuli. Unfortunately there are very badly behaved holomorphic functions on annuli, as we will be able to demonstrate after we finish proving the uniformization theorem, and I couldn't see an easy way to prove this extension. So we will make the extra assumption that f has a continuous extension.

d. By considering $\operatorname{Re} \ln f$, show that there must exist $u : A_R \to (0, S)$ that satisfies the Dirichlet problem

$$\Delta u = 0 \text{ on } A_R, \ u = 0 \text{ on } |z| = 1, \ u = \ln S \text{ on } |z| = R.$$

e. Why can there be at most one solution to this equation?

f. Find a radial solution so this equation, that is, a function that only depends on $|z|$.

g. Using Question 6.1 b, compute f. What conditions on R and S are required for this to be a well-defined function.