Funktionentheorie II – Exercise Set 5

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Question 5.1. Compute d' of the following real-valued functions and thereby show they are harmonic:

a. $u(z) = \ln|z| : \mathbb{C}^\times \to \mathbb{R}$,

b. more generally $v = \ln|f|$ for f a holomorphic function on X without zeroes.

They are the real parts of which holomorphic functions?

Question 5.2. Let $f : X \to Y$ be a holomorphic map between Riemann surfaces and $u \in \mathcal{H}(Y)$ harmonic. Show the composition is also harmonic.

Question 5.3. Find a Möbius transformation Φ that maps the open unit disc $\{z \in \mathbb{C} | \, |z| < 1\}$ to the upper half plane $\{z \in \mathbb{C} | \operatorname{Im} z > 0\}$ such that $\Phi(0) = i$.

Question 5.4. Consider the operation $*$ on 1-form called *Hodge star*:

$$*(\omega_1 \, dz + \omega_2 \, d\overline{z}) := i(-\omega_1 \, dz + \omega_2 \, d\overline{z}).$$

a. If the differential form is written in terms of dx and dy, compute the effect of $*$.

b. Show that a twice differentiable function h is harmonic when $*dh$ is closed.

c. Show that the function f defined by $df := dh + i * dh$ is holomorphic.

Question 5.5. In this question we will prove Dolbeault's Lemma: for every smooth function $g : \mathbb{C} \to \mathbb{C}$ with compact support, there is a smooth function f that solves

$$\frac{\partial f}{\partial \overline{z}} = g.$$

a. Let

$$f(z) := \frac{1}{2\pi} \int_{\mathbb{C}} \frac{g(\zeta)}{\zeta - z} \, d\zeta \wedge d\overline{\zeta}.$$

By applying the coordinate transformations $\zeta = z + w$ and $w = re^{i\theta}$ show that this is well-defined and that we are permitted to differentiate under the integral sign. Hence show that f is smooth.

b. Argue that

$$\frac{\partial f}{\partial \overline{z}} = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{C} \setminus B(0,\epsilon)} \frac{g(z+w)}{w} \, dw \wedge d\overline{w}$$

c. Compute the exterior derivative of $w^{-1} g(z+w)\, dw$. Thus prove that

$$\frac{\partial f}{\partial \overline{z}} = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\partial B(0,\epsilon)} w^{-1} g(z+w)\, dw$$

d. Conclude the proof of the lemma.