Funktionentheorie II – Exercise Set 4

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Questions marked with * are optional.

Question 4.1 (*). Is $f(z) = (z-1)^{-1}(z+1)^{-1}$ a covering map from $\hat{\mathbb{C}}$ to itself? Which choices of $a, b, c \in \mathbb{C}$ make $g(z) = (z-a)(z-b)^{-1}(z-c)^{-1}$ a covering map?

Question 4.2. Let X, Y be topological spaces, Y be connected, and $\pi : X \to Y$ a covering map so that $\pi^{-1}[\{y\}]$ is finite for every $y \in Y$. Then there exists $n \in \mathbb{N}$ so that $\#\pi^{-1}[\{y\}] = n$ for all $y \in Y$. This shows that the *degree* of a covering map is well defined. We say that π is an n-fold covering map.

Question 4.3. Prove that a holomorphic 1-fold covering map is a biholomorphism of Riemann surfaces.

Question 4.4. Consider a complex torus $\mathbb{C}/\Gamma$ for $\Gamma = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and the closed path $\alpha : [0,1] \to \mathbb{C}/\Gamma$ given by $\alpha(t) = t\omega_1 + \Gamma$. Produce any two lifts $\beta_i : [0,1] \to \mathbb{C}$ of α. Describe the relationship between the two lifts and between the endpoints of each lift.

Now consider now the closed path $\delta : [0,1] \to \mathbb{C}/\Gamma$ given by $\delta(t) = 2t\omega_1 + \Gamma$. Describe the lift of this map. How does it differ from the lifts in the previous part?

Question 4.5. Work through the construction of the universal cover in Theorem 1.45 for a complex torus. Explain the correspondence between curves on $\mathbb{C}/\Gamma$ (with fixed basepoint) up to homotopy and points of $\mathbb{C}$.

Question 4.6. Let X and Y be topological spaces and $\pi : X \to Y$ be a covering map. State what it means for π to be regular. If X is simply connected, then prove that π is regular.
[*Hint.* Deck transformations of π are lifts of π with respect to π. You might need to read this sentence twice.]

Question 4.7. Let X be a Riemann Surface whose universal covering $\pi : \tilde{X} \to X$ has finite degree. Let $\sigma : Y \to X$ be another holomorphic covering. Show that σ also finite degree and that its degree divides n.

Question 4.8 (*Follow-up to question about deducing topology from residues and differential forms). Consider $X = \mathbb{C}^{\times}$, the holomorphic differential $\eta = z^{-1}dz$, and α_i closed paths in X with the same endpoints. Use the residue theorem to give a critereon for when α_1 is homotopic to α_2. Thus deduce that the fundamental group of $\mathbb{C}^\times$ is $\mathbb{Z}$.