Funktionentheorie II – Exercise Set 3

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Questions marked with * are optional.

Question 3.1. Prove the product rule for differential forms: Let X be a Riemann surface, $f \in C^\infty(X)$, and $\omega \in \Omega^1(X)$. Then $d(f\omega) = df \wedge \omega + f\, d\omega$.

Question 3.2. Suppose $\omega \in \mathcal{A}^{1,0}(X)$. Show that ω is holomorphic if and only if it is closed.

Question 3.3. Consider $X = \mathbb{C}^\times$ with the global chart $z = x + iy$, ie $\mathrm{id}_{\mathbb{C}^\times}$, the smooth 1-form $\omega \in \Omega^1(X)$ that is given by

$$\omega = -\frac{y}{|z|^2}\, dx + \frac{x}{|z|^2}\, dy,$$

and the closed smooth curve $\gamma : [0, 2\pi] \to \mathbb{C}^\times$, $t \mapsto e^{it}$, which parameterises $\mathbb{S}^1$. Show that ω is closed. Compute $\int_\gamma \omega$ and conclude that ω is not exact.

Question 3.4. Consider the Riemann sphere $\hat{\mathbb{C}}$. Does there exist a meromorphic 1-form ω with a simple pole and no other poles?
[*Hint.* Calculate its residue in two ways.]
(*) Generalise this. What can be said about the residues of a meromorphic 1-form on any compact Riemann surface X? What about non-compact Riemann surfaces?

Question 3.5. Let $\pi : X \to Y$ be a covering map and suppose that $\#\pi^{-1}[\{y\}]$ is finite for all $y \in Y$. Prove that it is the same number for every point.
[*Hint.* In the notation of Definition 1.27, if $\tilde{y} \in V(y)$ show that $\#\pi^{-1}[\{\tilde{y}\}] = \#\pi^{-1}[\{y\}]$.]

(*) Now, suppose instead that $\#\pi^{-1}[\{y_0\}]$ is finite for some point $y_0 \in Y$. Does the conclusion still hold?

Question 3.6. Let $\Gamma \subset \mathbb{C}$ be a maximal lattice, $X = \mathbb{C}/\Gamma$ a complex torus and $\pi : \mathbb{C} \to X$ the cannonical projection. Further, let ℓ be the smallest distance between elements of the lattice. For every point $z \in \mathbb{C}$, define $B_z := B(z, \frac{1}{2}\ell) \subset \mathbb{C}$ to be the open disc around z with radius $\frac{1}{2}\ell$.

a. (*) Show that $\mathcal{A} = \{(\pi(B_z), (\pi|_{B_z})^{-1}) | z \in \mathbb{C}\}$ is an atlas for X.

b. (*) Show that π is a holomorphic map between Riemann surfaces.

Consider the map $\delta : X \to X$ defined by $z + \Gamma \mapsto 2z + \Gamma$.

c. Show that δ is a well-defined holomorphic map.

d. Is δ a covering map? If so, compute the *degree* $\#\delta^{-1}[\{0\}]$.