Funktionentheorie II – Exercise Set 2

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Questions marked with * are optional.

Question 2.1 (Identity Theorem.). Let $f, g : X \to Y$ be two holomorphic maps between Riemann surfaces, and suppose that the set $A = \{x \in X \mid f(x) = g(x)\}$ has an accumulation point in X. Then $f = g$ holds.

[*Hint.* Use the identity theorem from *Funktionentheorie I.* Show that the set of accumulation points of A is contained in A^o.]

Question 2.2. State the defintion of a discrete set. Suppose that $f : X \to Y$ is a non-constant, holomorphic map and $y \in Y$. Then $f^{-1}[\{y\}]$ is a discrete subset of X. If X is compact, show it is finite. What does this imply about the set of zeroes and poles of a meromorphic function?

Question 2.3 (Maximum and Minimum Principle for holomorphic functions). Let $f : X \to \mathbb{C}$ be a holomorphic function. Then $|f|$ does not attain a strict local maximum, and if it attains a strict local minimum at some $x_0 \in X$, then $f(x_0) = 0$ holds.
[*Hint.* Use the analogous statement from *Funktionentheorie I.*]

Question 2.4. Suppose that X is *compact.* Then any holomorphic map $f : X \to \mathbb{C}$ is constant.

Question 2.5. Suppose that X, Y are both *compact.* Either prove or give a counterexample for the following statement: Any holomorphic map $f : X \to Y$ is constant.

Question 2.6 (*Riemann's theorem on removable singularities). Let $x_0 \in X$ and $f : X \setminus \{x_0\} \to Y$ be a holomorphic map. Suppose that one of the following two statements holds:

1. f can be extended to a continuous function $g : X \to Y$.
2. $Y = \mathbb{C}$ and there exists a neighbourhood U of x_0 in X so that the function $\left| f|_{U\setminus\{x_0\}} \right|$ is bounded.

Then f can be extended at x_0 to a holomorphic map $g : X \to Y$.

Question 2.7. Consider the usual atlas on the Riemann sphere $\{(U_1, \mathrm{id}), (U_2, z \mapsto$

$z^{-1})\}$. Let ω be a differential 1-form with the expression

$$\omega|_{U_1} = (z + \overline{z})\, dz + z\overline{z}^{-1}\, d\overline{z}.$$

That is, $\omega_{1,1} = z + \overline{z}$ and $\omega_{1,2} = z\overline{z}^{-1}$. Use the transformation formula to give an expression for ω on U_2. That is, find $\omega_{2,1}$ and $\omega_{2,2}$.

Question 2.8. Take two differential 1-forms $\omega, \eta \in \Omega^1(X)$ on a Riemann surface X with atlas $\{(U_\alpha, z_\alpha)\}$. They of course have local expressions $\omega|_{U_\alpha} = \omega_{\alpha,1}\, dz + \omega_{\alpha,2}\, d\overline{z}$ and $\eta|_{U_\alpha} = \eta_{\alpha,1}\, dz + \eta_{\alpha,2}\, d\overline{z}$. On every open set U_α in the atlas, define

$$\psi_\alpha := \omega_{\alpha,1}\eta_{\alpha,2} - \omega_{\alpha,2}\eta_{\alpha,1}$$

Show that these functions together define a 2-form ψ on X. In particular, you should show that they satisfy the transformation formula.