Funktionentheorie II – Exercise Set 13

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Dear Students, again there are a lot of question this week. I will try to cover 13.2, 13.3, and 13.6a-d in tutorial.

Question 13.1. This question was in the lectures, but it is the sort of exercise you should be able to do without looking up the notes. Try it for yourself, but ask if any part does not make sense to you.

Apply Serre duality and the Riemann-Roch theorem to conclude the following about any canonical divisor K.

a. $\deg K = 2g - 2$, where g is the genus,

b. $\dim H^0(X, \mathscr{O}_K) = g$.

Question 13.2 (Converse to Corollary 3.63). Let X be a compact Riemann surface with genus $g = 1$.

a. Prove that there is a non-vanishing holomorphic differential ω.

Let $\tilde{X}$ be the universal cover of X and choose a point $x_0 \in \tilde{X}$. Define a map $F : \tilde{X} \to \mathbb{C}$ by

$$F(x) = \int_{x_0}^{x} \omega.$$

b. Argue that F is a well-defined holomorphic function.

c. Further, show that F is a biholomorphism.

d. Hence prove that X is a torus $\mathbb{C}/L$.

Question 13.3. In this question we introduce the Abel map of a compact Riemann surface X. We have seen above that the space of global holomorphic differential forms $\Omega^1(X) = H^0(X, \mathscr{O}_K)$ is g dimensional. Therefore take a basis $\omega_1, \ldots, \omega_g$. Recall that the universal cover X can be defined as an equivalence class of paths. Choose a base point $x_0 \in X$ and define $\tilde{\mathcal{A}}_{x_0} : \tilde{X} \to \mathbb{C}^g$ by

$$\tilde{\mathcal{A}}_{x_0}(\gamma) = \begin{pmatrix} \int_\gamma \omega_1 \\ \vdots \\ \int_\gamma \omega_g \end{pmatrix}.$$

However, we would like a map defined on the Riemann surface X itself. The obstacle is that there are many paths from the base point x_0 to a given point x. However, any two paths differ by a loop in X. Therefore we define the *periods*

$$L = \{\tilde{\mathcal{A}}_{x_0}(\gamma) \mid \gamma \text{ is a loop in } X\},$$

the *Jacobian* $\mathrm{Jac}(X) := \mathbb{C}^g/L$ and the Abel map

$$\mathcal{A}_{x_0}(x) = \tilde{\mathcal{A}}_{x_0}(\gamma) + L \in \mathrm{Jac}(X), \quad \text{for any path } \gamma \text{ from } x_0 \text{ to } x.$$

a. Extend $\mathcal{A}_{x_0}$ from a map from X to $\mathrm{Jac}(X)$ to a homomorphism from $\mathrm{Div}(X)$ to $\mathrm{Jac}(X)$.

b. Show that for divisors with degree zero, this map is independent of the choice of base point.

c. (*) Take any meromorphic function f on X. We can view this as a holomorphic map from X to $\hat{\mathbb{C}}$. For any point $p \in \hat{\mathbb{C}}$, define the preimage divisor as

$$f^{-1}(p) = \sum_{f(x)=p} \mathrm{ord}_x\, df \cdot x.$$

This is simply the preimages of p counted with multiplicity. Consider $\phi : \hat{\mathbb{C}} \to \mathrm{Jac}(X)$ given by the composition $\phi(p) = \mathcal{A}(f^{-1}(p))$. Prove this function is constant.

Hint. Lift ϕ to a function from $\hat{\mathbb{C}}$ to $\mathbb{C}^g$.

d. By writing the divisor of f in terms of preimage divisors, show that the image of a principal divisor is zero under the Abel map: $\mathcal{A}((f)) = 0$.

Hence we see that two necessary conditions for a divisor D to be principal are that it is degree zero and that its image under $\mathcal{A}$ is zero.

Question 13.4 (Theta Functions *). By Theorem 1.40, we may now assume that $X = \mathbb{C}/(\mathbb{Z}1 \oplus \mathbb{Z}\tau)$ for $\mathrm{Im}\,\tau > 0$. On $\mathbb{C}$ define the *theta function*

$$\theta(z) = \sum_{k=-\infty}^{\infty} \exp\left[2\pi i \left(k + \frac{1}{2}\right) z\right] \exp\left[\pi i \left(k + \frac{1}{2}\right)^2 \tau\right].$$

Note: there are multiple conventions for the precise definition of this function. This one is ugly, but gives the result most directly.

a. Argue that this series converges absolutely and uniformly on compact subsets of $\mathbb{C}$. Hence it defines a holomorphic function.

b. Show that this function is quasiperiodic:

$$\theta(z+1) = \theta(z), \qquad \theta(z+\tau) = \exp\left[-\pi i(2z+\tau)\right]\theta(z),$$

and odd $\theta(z) = -\theta(z)$.

c. Consider the parallelogram $P = [0,1] + [0,1]\tau$. Because the zeroes of a holomorphic function are discrete, there must exist $z_0 \in \mathbb{C}$ such that θ does not vanish on $z_0 + \partial P$. Calculate directly

$$\int_{z_0+\partial P} \frac{\theta'(z)}{\theta(z)}\,\mathrm{d}z = 2\pi i.$$

d. Conclude that $\theta(z)$ has simple zeroes at the points of $L = \mathbb{Z} \oplus \mathbb{Z}\tau$ and nowhere else.

e. Demonstrate, for any L-periodic meromorphic function h,

$$\int_{z_0+\partial P} z\frac{h'(z)}{h(z)}\,\mathrm{d}z \in \mathbb{Z} \oplus \mathbb{Z}\tau,$$

where z_0 is chosen so that no zeroes or poles of h lie on the boundary.

Question 13.5 (Abel's Theorem). Abel's theorem states that necessary and sufficient conditions for a divisor D to be principal are that it is degree zero and that its image under $\mathcal{A}$ is zero. We have already proved necessity. The standard proof of sufficiency requires existence theorems about meromorphic differentials with certain poles. We will prove this theorem only for complex tori, using theta functions. resume,,

e. Explain how Abel's theorem applies to the Riemann sphere $\hat{\mathbb{C}}$ and how you already knew this fact.

f. Let X be a compact, genus 1 Riemann surface. In question 13.2c we saw that $X = \mathbb{C}/L'$. Using the fact that dz is a nonvanishing holomorphic differential on X, what is the relationship between L' and the lattice of periods defined in this question?

g. Suppose that a divisor D is degree zero. Hence we may write $D = \sum_i 1 \cdot p_i - 1 \cdot q_i$ for points p_i and q_i. Choose representatives $\tilde{p}_i, \tilde{q}_i \in \mathbb{C}$ that lie over p_i, q_i. Define a meromorphic function on $\mathbb{C}$ by

$$f(z) = \prod_i \frac{\theta(z - \tilde{p}_i)}{\theta(z - \tilde{q}_i)}.$$

Where are the zeroes and poles of f?

h. Using questions 13.3b and e, prove that if $\mathcal{A}(D) = 0$, then f defines a function on X with $(f) = D$.

Question 13.6 (A more general definition of Jacobians). We have seen that one way to view a divisor is as a function $X \to \mathbb{Z}$ with discrete support. Thus we have the sheaf of divisors Div. We also have morphism from $\mathscr{M}^\times$, the meromorphic functions (excluding 0) with multiplication as the group operation, to Div given by taking the divisor of a function.

a. Why is $\mathscr{O}^\times$ the kernel of this morphism?

Therefore we have an exact sequence $0 \to \mathscr{O}^\times \to \mathscr{M}^\times \to \mathrm{Div} \to 0$. One can show that both $H^1(X, \mathscr{M}^\times)$ and $H^1(X, \mathrm{Div})$ are zero. Hence the corresponding long exact sequence of cohomology is

$$0 \to H^0(X, \mathscr{O}^\times) \to H^0(X, \mathscr{M}^\times) \to H^0(X, \mathrm{Div}) \to H^1(X, \mathscr{O}^\times) \to 0.$$

b. Argue that $H^0(X, \mathscr{M}^\times)/H^0(X, \mathscr{O}^\times)$ is the space of principal divsors $\mathrm{PDiv}(X)$.

c. Hence show that $Div(X)/PDiv(X) \cong H^1(X, \mathscr{O}^\times)$.

Thus $H^1(X, \mathscr{O}^\times)$ is a measure of how many divisors on X do not come from meromorphic functions. This space has a special name, it is called the *Picard group* of X. Recall now the exponential exact sequence $0 \to \mathbb{Z} \to \mathscr{O} \to \mathscr{O}^\times \to 0$ that we have seen several times before. The non-trivial morphism is $f \mapsto \exp 2\pi i f$.

d. Prove that there is an injective homomorphism

$$\frac{H^1(X, \mathscr{O})}{H^1(X, \mathbb{Z})} \to H^1(X, \mathscr{O}^\times).$$

e. (*) Argue that $H^1(X, \mathscr{O})/H^1(X, \mathbb{Z})$ when viewed as a subspace of $\mathrm{Div}(X)/\mathrm{PDiv}(X)$ consists of exactly the degree zero divisors.

f. (*) In the case of a compact Riemann surface X, use Serre duality to demonstrate that $H^1(X, \mathscr{O})/H^1(X, \mathbb{Z})$ is $\Omega(X)^*/L$, the Jacobian of X.