Funktionentheorie II – Exercise Set 12

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Question 12.1. Make a table to summarise the results we have proved about H^1 for different sheaves and different spaces. You might want to include both specific spaces, such as $\mathbb{C}^\times$, and categories of spaces, such as simply connected.

Question 12.2. In this question we show directly the isomorphism in Theorem 3.51(a). Recall this sequence

$$0 \to \mathscr{O} \to \mathscr{E} \to \mathscr{E}^{(0,1)} \to 0$$

from Beispiel 3.16c, which we proved was exact in a previous exercise.

a. Write out the corresponding long exact sequence.

b. Therefore conclude Theorem 3.51(a).

c. We now examine which 1-forms are exact. Find a cover $\mathfrak{U}$ of X suitable to apply Dolbeault's Lemma to each chart.

d. Take any element $\omega \in \mathscr{E}^{(0,1)}(X)$ and apply the lemma to construct a cochain f in $C^0(\mathfrak{U}, \mathscr{E})$ with the property that $d''f = \omega$.

e. Conclude that ω is exact if and only if $[\delta(f)] \in H^1(\mathfrak{U}, \mathscr{O})$ is zero.

Question 12.3. Let X be a Riemann surface, $f, g \in \mathcal{M}(X) \setminus \{0\}$ be meromorphic functions (not identically zero) and $\omega \in \mathcal{M}^{(1,0)}(X) \setminus \{0\}$ a meromorphic 1-form. Prove the following standard facts for divisors.

$$(fg) = (f) + (g), \qquad (1/f) = -(f), \qquad (f\omega) = (f) + (\omega).$$

State what it means for a divisor to be canonical, and prove that all canonical divisors are equivalent.

Question 12.4. Let $X = \hat{\mathbb{C}}$.

a. Compute the divisors of the following functions:

$$f(z) = \frac{1}{z^2}, \qquad g(z) = z^2 - 1, \qquad h(z) = \frac{z^2-1}{z^2}.$$

b. Write down two canonical divisors. Show directly that they are linearly equivalent by finding an appropriate meromorphic function.

c. Give all functions in $\mathscr{O}_D$ for the following divisors D:

$$1 \cdot 0, \qquad 2 \cdot 0, \qquad 2 \cdot 0 - 1 \cdot \infty, \qquad (g).$$

Question 12.5. Consider the set $X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = x(x-1)(x-2)\}$.

a. (*) We saw in Question 1.2 that the graphs of biholomorphic functions are Riemann surfaces. Extend this to show X is a Riemann surface, with coordinate charts given by the restrictions of $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$.

b. (*) Show that the restriction of a holomorphic function $\mathbb{C}^2 \to \mathbb{C}$ is a holomorphic function on X.

c. Compute the divisors of the functions π_1 and π_2 on X.

d. Compute the divisors of the differentials $d\pi_1$ and $d\pi_2$.

e. Show that there is a holomorphic map $\sigma : X \to X$ which is involutive (i.e. $\sigma \circ \sigma = \mathrm{id}_X$) and which satisfies $\pi_1 \circ \sigma = \pi_1$ and $\pi_2 \circ \sigma = -\pi_2$.