## Funktionentheorie II – Exercise Set 11

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**Question 11.1.** Let $X = B(0, R)$ for $0 < R \leq \infty$ and $g \in \mathscr{E}(X)$. Show using Dolbeault's Lemma that there exists $f \in \mathscr{E}(X)$ that solves Laplace's equation $\Delta f = g$.

**Question 11.2.** Prove for any compact Riemann surface $X$ that $H^1(X, \mathbb{C})$ is finite dimensional. Generalise this result.

**Question 11.3.** Recall the symbols for the following sheaves on a Riemann surface $X$: $\mathscr{E}^{(1,0)}$ and $\mathscr{E}^{(0,1)}$ are sheaves of smooth differential 1-forms of type $(1, 0)$ and $(0, 1)$ respectively; $\Omega^1 \subset \mathscr{E}^{(1,0)}$ is the sheaf of holomorphic differential 1-forms of type $(1, 0)$; $\Omega^2$ is the sheaf of holomorphic 2-forms. Prove the following two sequences are exact (cf Beispiel 3.16)

$$0 \to \mathscr{O} \to \mathscr{E} \xrightarrow{d''} \mathscr{E}^{(0,1)} \to 0,$$

$$0 \to \mathbb{C} \to \mathscr{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to 0.$$

**Question 11.4.** Let $\phi : \mathscr{F} \to \mathscr{G}$ be a surjective morphism of sheaves. In this question we will define a map $\delta^*$ from $H^0(X, \mathscr{G})$ to $H^1(X, \ker \phi)$ called the connecting homomorphism.

a. Argue why for every point $p \in X$ there exists an open neighbourhood $U_p$ and a section $f_p \in \mathscr{F}(U_p)$ such that $\phi(f_p) = g|_{U_p}$.

b. The sections $(f_p)$ define a 0-cochain $f$ with respect to the cover $\{U_p\}$. Define $h = \delta^0(f)$. Show that $\phi(h) = 0$.

   (Note that any morphism of sheaves gives a map of cochains by acting on each element of the cochain.)

c. Explain why $h$ is a cocycle of the sheaf $\ker \phi$ but is not necessarily a coboundary of it.

d. (*) We say that $\delta^*(g)$ is the class in $H^1(X, \ker \phi)$ given by $h$. Show that this does not depend on the choice of neighbourhoods $U_p$ or sections $f_p$.

e. Show that $\delta^*$ is a homomorphism.