Funktionentheorie II – Exercise Set 10

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Question 10.1. Prove that the cocyle condition (Beispiel 3.19) implies that $\tilde{g} = (g_{ij}) \in C^1(\mathfrak{U}, \mathscr{F})$ is a 1-cocycle only if $f_{ii} = 0$ and $f_{ij} = -f_{ji}$. Hence what information is sufficient to determine a cocycle?

Question 10.2. Suppose that $X = \mathbb{C}^\times$ and the take the cover to be $\mathfrak{U} = \{\mathbb{C} \setminus \mathbb{R}_{\geq 0}, \mathbb{C} \setminus \mathbb{R}_{\leq 0}\}$. Calculate $H^1(\mathfrak{U}, \mathbb{Z})$ and $H^1(\mathfrak{U}, \mathbb{C})$. Hence what can we conclude about the cohomology of X?

Question 10.3. Suppose that X is a simply connected Riemann surface and $\mathfrak{U}$ is an open cover. Show the following:

a. $H^1(\mathfrak{U}, \mathbb{C}) = 0$.

b. $H^1(\mathfrak{U}, \mathbb{Z}) = 0$.

Question 10.4. In this question we show the difficulty of computing the cohomology of the sheaf $\mathscr{O}$. Let $X = \mathbb{CP}^1$ and take $\mathfrak{U} = \{U_1 = \mathbb{C}, U_2 = \mathbb{C}^\times \cup \{\infty\}\}$ as the cover.

a. Show that

$$\tilde{g} = (g_{11}, g_{12}, g_{21}, g_{22}) \in \mathscr{O}(U_1) \times \mathscr{O}(U_1 \cap U_2)^2 \times \mathscr{O}(U_2) = C^1(\mathfrak{U}, \mathscr{O})$$

is a cocycle exactly when $g_{11} = g_{22} = 0$ and $g_{12} = -g_{21}$. (Compare 10.1)

b. Show that $\tilde{g} \in C^1(\mathfrak{U}, \mathscr{O})$ is a boundary exactly when there are holomorphic functions f_i on U_i such that $g_{12} = f_2 - f_1$.

c. Prove that every 1-cocycle is a coboundary. *Hint.* Use a Laurent expansion of g_{12}.

Now consider again $X = \mathbb{C}^\times$ from Question 10.2. Can you apply these ideas to X? What are the obstacles?

Question 10.5. Let X be a topological space and recall the skyscraper sheaf at $q \in X$ valued in G from Exercise Sheet 8. Compute the cohomology of this sheaf $H^1(X, i_q(G))$ directly from the definition.

Question 10.6 (**). This question has two stars because it's super-optional. We'll discuss some points of it in two weeks time, to give you time to think about it. I think parts (a)-(i) are reasonable. If the proofs in the later parts are too hard for a general surface, do them with just for the examples from the earlier parts.

In this question we sketch the relationship between cohomology with constant sheaves and simplicial cohomology. In particular we show its combinatorial nature. This question ended up being very long and beyond the scope of the course; sorry. Also I haven't checked all the details, so you might have to make some corrections. I'm fairly confident it works for vector spaces (so for the sheaf $\mathbb{C}$) but less confident about it for modules over a principal ideal domain (eg $\mathbb{Z}$).

We say that a cover $\mathfrak{U}$ is *combinatorial* when it is an open, finite cover of an oriented surface such that the sets U_i, $U_i \cap U_j$, and $U_i \cap U_j \cap U_k$ are all contractible (ie homeomorphic to a disc), and all distinct higher intersections are empty (ie there is no point that lies in four distinct sets of the cover).

a. Give a combinatorial cover for $\mathbb{CP}^1$.

Consider the graph with one vertex for each open set U_i. Two distinct vertices are connected by an edge if $U_i \cap U_j$ is not empty, and three distinct vertices span a face if $U_i \cap U_j \cap U_k$ is not empty.

b. Draw this graph for your cover in part (a).

c. Give a combinatorial cover for $\mathbb{C}^\times$ and draw its graph.

d. Under the assumption that every Riemann surface is triangulable by finitely many triangles, prove that a combinatorial cover always exists.

e. Show that the number of edges is greater than the number of faces.

f. Establish that the surface is compact if and only if every edge adjoins two faces.

Let $\mathscr{R}$ be a sheaf of locally constant functions valued in a principal ideal domain R (ie $\mathbb{C}$ and $\mathbb{Z}$ are both of this form).

g. Explain why a 0-cochain is equivalent to the choice of an element v_i of G for each vertex.

h. Using 10.1, explain why a 1-cocyle is described by the choice of an element e_{ij} of G and an orientation for each edge.

i. Restate the cocycle relation in this situation.

Now, represent δ^1 as a matrix from $R^E \supset Z^1$ to $R^F \cong Z^2$ with entries $\{0, \pm 1\}$. Use the orientation of surface to fix an orientation for each face. If an edge adjoins two faces, then there is both a 1 and a -1 in the corresponding column of δ^1.

j. Argue that if every edge adjoins two faces, then knowing the value of $\delta^1((e_{ij}))$ on every face except one determines its value on the remaining face.

k. Show that for a non-compact Riemann surface the matrix of δ^1 can be put into row echelon form with rank F. Moreover there exist invertible matrices S, T so that
$$S\delta^1 T = [I_F \,|\, 0].$$
Hence $Z^1 \cong R^{E-F}$ by the structure theorem of finitely generated modules over a principal ideal domain and $H^2 := C^2/\operatorname{img}\delta^1 = 0$.

Next, we consider must consider the module Z^1 quotiented by the image $\delta^0(C^0)$.

l. An oriented edge has a 'head' and a 'tail' vertex. Show that δ^0 assigns it the value of 'head' $-$ 'tail'.

m. Show that δ^0 maps a 0-cochain to $0 \in C^1$ if and only if it has the same value at every vertex.

n. Again, write δ^0 as a matrix between $C^0 = R^{V-1} \times R\langle(1, \ldots, 1)\rangle$ and $Z^1 \cong R^{E-F}$. Prove $H^1 \cong R^{E-F-V+1}$.

Finally, complete the proof in the case of a compact Riemann surface. You should find that $H^1 \cong R^{E-F-V}$ and $H^2 \cong R$.