Funktionentheorie II – Exercise Set 1

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Questions marked with * are optional.

Question 1.1. We saw in the lecture that $\mathbb{C}$ is a Riemann surface with the chart $(\mathbb{C}, \mathrm{id}_\mathbb{C})$. Consider the chart $(\mathbb{C}, \phi)$ where $\phi(z) = \overline{z}$. Are these two charts holomorphically compatible? Are they compatible as (real) smooth charts?

Question 1.2. Let (U, ϕ) be a chart of X and f a biholomorphic function from $\phi[U]$ to an open set $W \subset \mathbb{C}$. Verify that $(U, f \circ \phi)$ is a chart of X that is holomorphically compatible with (U, ϕ).

Let $X = \mathrm{graph} f = \{(z, f(z)) \in \mathbb{C}^2 \mid z \in \phi[U]\}$ and $\pi_1, \pi_2 : X \to \mathbb{C}$ projections onto each component of $\mathbb{C}^2$. Show that $\{(X, \pi_1), (X, \pi_2)\}$ is a holomorphic atlas for X.

Question 1.3. Consider the map $F : \mathbb{C} \to \hat{\mathbb{C}}$ that is given by $z \mapsto \frac{z}{z-1}$ using the convention that $1/0 = \infty$. Using the two coordinate charts for $\hat{\mathbb{C}}$ from Beispiel 1.5(b), write down this map in coordinate charts in a way that does not use $1/0 = \infty$. Is this a holomorphic map between Riemann surfaces in the sense of Definition 1.7?
[*Hint.* Use the coordinate charts $(\mathbb{C} \setminus \{1\}, \mathrm{id})$ and $(\mathbb{C} \setminus \{0\}, \mathrm{id})$ on $\mathbb{C}$.]

Question 1.4 (*). Show that $\hat{\mathbb{C}}$ is homeomorphic to $\mathbb{S}^2$.
[*Hint.* Stereographic projection gives a homeomorphism between $\mathbb{S}^2 \setminus \{(0,0,1)\}$ and the plane. Look up and use this formula; you do not need to rederive it.]

Question 1.5. Look up the definition of complex projective space $\mathbb{C}\mathrm{P}^1$. Describe it as a Riemann surface. That is, describe the points of this space and a holomorphic atlas.

Question 1.6. Show that $\hat{\mathbb{C}}$ is biholomorphic to $\mathbb{C}\mathrm{P}^1$.