

**Preliminary Exercises**

**38. Divergence and rotation.**

(a) Let  $\Omega \subset \mathbb{R}^2$  be open and  $L$  a (non-elliptic) differential operator on  $\Omega$ , defined by

$$Lu := \partial_1(\partial_2 u) - \partial_2(\partial_1 u).$$

Show that every  $u \in W^{1,2}(\Omega)$  is a solution of  $Lu = 0$  in the weak sense.

(b) Let  $\Omega \subset \mathbb{R}^n$  be open. We say that a vector field  $f = (f_1, \dots, f_n) : \Omega \rightarrow \mathbb{R}^n$  with  $f_k \in L^2(\Omega)$ ,  $k \in \{1, \dots, n\}$  is a weak solution the differential equation  $\nabla \cdot f = 0$  when

$$\int_{\Omega} (f \cdot \nabla \phi) d\mu = 0 \quad \text{for all } \phi \in W_0^{1,2}(\Omega).$$

Now set  $n = 3$ . The *curl* (also called the *rotation*) of a vector field  $u = (u_1, u_2, u_3) : \Omega \rightarrow \mathbb{R}^3$  with  $u_k \in W^{1,2}(\Omega)$ ,  $k \in \{1, 2, 3\}$  is defined to be

$$\nabla \times u := (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1),$$

in analogy to the cross product  $\times$ .

Show that the curl  $f := \nabla \times u$  of  $u$  is a weak solution of  $\nabla \cdot f = 0$ .

**39. On Friedrich's Theorem on the interior.**

Consider the real, open intervals  $I := (-2, 2)$  and  $J := (-1, 1)$ . We choose a function  $a \in L^\infty(I) \setminus W^{1,2}(J)$  with  $a \geq 1$ , and let

$$u : I \rightarrow \mathbb{R}, \quad u(t) := \int_0^t \frac{1}{a(x)} dx.$$

(a) Show that  $u \in W^{1,2}(I)$  and  $u \notin W^{2,2}(J)$ .

(b) Show that  $u$  is a weak solution of  $(au')' = 0$  on  $I$ .

(c) Why does this not contradict Friedrich's theorem on the interior?

**In Class Exercises**

**40. On the Cacciopoli inequality at the boundary.**

Complete the proof of the Cacciopoli inequality at the boundary (Theorem 4.7) from the lecture notes by adapting the proof of Theorem 4.5.