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Partial Differential Equations
Exercise sheet 10

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Preparatory Exercises

35. The dual to $W_0^{1,2}$.

At the beginning of Chapter 4 we are introduced to the dual space $W_0^{1,2}(\Omega)^*$. In this question we explore this a little more.

(a) Show that the formula for $\langle g + \nabla \cdot f, v\rangle$ given at the start of Chapter 4 does indeed define a distribution for $v \in C_0^\infty(\Omega)$. Why does it extend to $W_0^{1,2}(\Omega)$?

(b) Give an example of two different choices g, f, g', f' that give the same distribution.

(c) In Chapter 3 we saw that $W_0^{1,2}(\Omega)$ was a Hilbert space. Recall the definition of its inner product. What is the relation between this inner product and the norm on $W^{1,2}$.

(d) Optional: Use the Riesz representation theorem to show that all extending distributions have this form.

36. On the weak solutions of elliptic differential equations.

Let $\Omega \subset \mathbb{R}^n$ be open, $u \in W^{1,2}(\Omega)$, $f \in W_0^{1,2}(\Omega)^*$, and L be an elliptic differential operator in the sense of Definition 4.1.

(a) State what it means for u to be a weak solution of $Lu \geq f$.

(b) Show that the following is a distribution:

$$C_0^\infty(\Omega) \ni \phi \mapsto -\mathcal{L}(u, \phi).$$

(c) Suppose that u is a weak solution of both

$$Lu \geq f \quad \text{and} \quad Lu \leq f, \tag{$*$}$$

in the sense of Definition 4.1. Show for all $v \in W_0^{1,2}(\Omega)$ that $-\mathcal{L}(u, v) = \langle f, v\rangle$. Note: there is no requirement for v to be non-negative.

(d) How should we interpret Lu as a distribution, if it is not L^1_{loc}? Hence prove that if $(*)$ holds then $Lu = f$ in the sense of distributions.

(e) Suppose now that $u \in W^{1,2}_{\text{loc}}(\Omega)$, $f \in L^2_{\text{loc}}(\Omega)$ such that $\triangle u \geq f$ and $\triangle u \leq f$ hold in the weak sense. Show for all $\phi \in C_0^\infty(\Omega)$ that

$$\triangle(\phi u) = (\triangle \phi)u + 2\nabla\phi \cdot \nabla u + f\phi$$

holds in the sense of distributions.

In Class Exercise

37. Weak solutions of the Poisson equation.

In the following we demonstrate an example of functions $u, f \in C^0(\Omega)$ such that $\triangle u = f$ in the weak sense, but $u \notin C^2(\Omega)$. Let $\Omega = B(0, \frac{1}{2}) \subset \mathbb{R}^2$ and $u(x,y) := (x^2 - y^2) \log|\log(r)|$ with $r = (x^2 + y^2)^{1/2}$.

(a) Show that $u \in C^2(B(0, \frac{1}{2}) \backslash \{0\})$ and $\lim_{r \to 0} u(x,y) = 0$. In other words, u extends to a continuous function on $B(0, \frac{1}{2})$.

(b) Compute the following derivatives of u on $B(0, \frac{1}{2}) \backslash \{0\}$

$$\begin{aligned} \frac{\partial}{\partial x} u(x,y) &= 2x \log|\log(r)| + (x^3 - y^2 x) \frac{1}{r^2 \log(r)}, \\ \frac{\partial^2}{\partial x^2} u(x,y) &= 2 \log|\log(r)| + (5x^2 - y^2) \frac{1}{r^2 \log(r)} - (x^4 - x^2 y^2) \frac{2\log(r) + 1}{r^4 (\log(r))^2}. \end{aligned}$$

(c) Argue that $\frac{\partial^2}{\partial y^2} u(x,y) = -\frac{\partial^2}{\partial x^2} u(y,x)$ and hence

$$\triangle u = (x^2 - y^2) \left(\frac{4}{r^2 \log(r)} - \frac{1}{r^2 (\log(r))^2} \right).$$

Conclude therefore that $\lim_{r \to 0} \triangle u(x,y) = 0$.

(d) Let $f \in C(B(0, \frac{1}{2}))$ be the continuous extension of $\triangle u$ on $B(0, \frac{1}{2})$. Prove that $\triangle u = f$ weakly on $B(0, \frac{1}{2})$.