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Partial Differential Equations
Exercise sheet 9

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Preparatory Exercises

31. Another approach to Sobolev inequalities.

Sobolev inequalities compare the "size" of ∇u with that of u. Therefore we want to express u in terms of its gradient.

(a) Let Ω be bounded and $u \in C_0^\infty(\Omega) \subset C_0^\infty(\mathbb{R}^n)$ and take polar coordinates $(r, v) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ on $\mathbb{R}^n$. Show:

$$u(x) = -\frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} \int_0^\infty \partial_r(u(x + rv))\, \mathrm{d}r\, \mathrm{d}\sigma(v).$$

[Hint. First compute $-\int_0^\infty \partial_r(u(x + rv))\, \mathrm{d}r$.]

(b) Prove further that

$$u(x) = \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{\langle x - y, \nabla u(y)\rangle}{|y - x|^n}\, \mathrm{d}y \text{ and } |u(x)| \le \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|y - x|^{n-1}}\, \mathrm{d}y.$$

(c) Find a bound on u in terms of $\|\nabla u\|_p$ for $p > n$.

32. The Sobolev conjugate.

Suppose that for compactly supported smooth functions we have an inequality

$$\|u\|_q \le C\|\nabla u\|_p.$$

By considering the rescaled functions $u_\lambda(x) := u(\lambda x)$ show that this inequality is only possible for $q^{-1} = p^{-1} - n^{-1}$.

In Class Exercises

33. The Sobolev embedding theorem.

Show that $W^{1,1}((0,1)) \hookrightarrow C([0,1])$ is a continuous embedding.

[Hint. One needs to show that $\|u\|_\infty \le \|u\|_1 + \|u_1\|_1$ holds. Therefore define, for $(u, u_1) \in W^{1,1}((0,1))$, the function $U := \int_{x_0}^x u_1(t)\, \mathrm{d}t$ and prove: $U \in W^{1,1}((0,1)) \cap C([0,1])$ and $U - u \equiv \text{const}$. It then follows that $|u|$ obtains a minimum $x_0 \in [0,1]$. Finally, one can show $|u(x) - u(x_0)| \le \|u_1\|_1$ and estimate $\|u\|_\infty$ with the triangle inequality.]

34. The Garding inequality.

The Garding inequality, Equation (4.5) in the script, is needed to apply the Lax-Milgram theorem. Here we prove a special case. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain and $L : C_0^2(\Omega) \to C_0(\Omega)$ the elliptic operator

$$(Lu)(x) = -\mathrm{div}(A(x)\nabla u(x)) + c(x)u(x)$$

given in divergence form. Let $K > 0$ and $c(x) \ge K \;\; \forall x \in \Omega$. Show that L obeys the inequality

$$\langle Lu, u\rangle_{L^2(\Omega)} \ge C \cdot \|u\|^2_{W^{1,2}(\Omega)} \text{ (for a constant } C > 0).$$