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Partial Differential Equations
Exercise sheet 7

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Preparatory Exercises

23. The dual space of $L^p(\mathbb{R}^n)$.

Let $1 < p < \infty$ (we exclude $p = 1$ for this exercise). The Banach space $L^p(\mathbb{R}^n)$ has the norm

$$\|\cdot\| : L^p(\mathbb{R}^n) \to \mathbb{R}, \ \ f \mapsto \|f\|_p = \left(\int_{\mathbb{R}^n} |f|^p \mathrm{d}\mu\right)^{1/p}.$$

We will show that for q with $\frac{1}{p} + \frac{1}{q} = 1$ the map

$$j : L^q(\mathbb{R}^n) \to L^p(\mathbb{R}^n)' = \mathcal{L}(L^p(\mathbb{R}^n), \mathbb{R}), \ \ g \mapsto j(g) \text{ with } j(g)(f) = \int_{\mathbb{R}^n} fg \, \mathrm{d}\mu$$

is a linear isometry, i.e. $\|g\|_q = \|j(g)\|$ holds. One can then show that for $1 \leq p < \infty$ the dual space of $L^p(\mathbb{R}^n)$ is isometrically isomorphic to $L^q(\mathbb{R}^n)$.

(a) Show, with the help of the Hölders inequality that $j : L^q(\mathbb{R}^n) \to L^p(\mathbb{R}^n)'$ is Lipschitz continuous with Lipschitz constant $L \leq 1$.

(b) Given a function g, find a function f_g such that $|j(g)(f_g)| = \|f_g\|_p \cdot \|g\|_q$.

(c) Show that j is an isometry.

(d) Optional: Use the Radon-Nikodym theorem to prove that j is surjective.

(e) Finish the proof that $L^p(\mathbb{R}^n)'$ and $L^q(\mathbb{R}^n)$ are isometrically isomorphic.

(f) Optional: Extend this result to the case $p = 1$ and $q = \infty$.

(g) What is the connection to distributions and Proposition 3.22?

24. Sobolev Functions.

(a) Write the definition of a Sobolev space using distributions.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$x \mapsto \begin{cases} 1 + x & \text{für } -1 \leq x \leq 0 \\ 1 - x & \text{für } \ \ 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Describe the first derivative of the distribution $F : C_0^\infty(\mathbb{R}) \to \mathbb{R}, \ \phi \mapsto F(\phi) = \int_{\mathbb{R}} f(x)\phi(x)\mathrm{d}x$.

(ii) Show that the second derivative of the distribution $F(\phi) = \int_{\mathbb{R}} f(x)\phi(x)\mathrm{d}x$ is a linear combination of Dirac distributions.

(iii) Show: $f \in W^{1,1}(\mathbb{R})$, but $f \notin W^{2,1}(\mathbb{R})$.

(c) Let $\Omega = \mathbb{R}^n$ and $u \in W^{2,1}_{\text{loc}}(\mathbb{R}^n)$ so that $\partial^\alpha u = 0$ for all α with $|\alpha| = 2$ in the weak sense. Show that u is affine, i.e. $u(x) = a \cdot x + b$ *a.e.* with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

[Hint. Proposition 3.22.]

(d) Let $\Omega = B(0, 0.5) \subset \mathbb{R}^2$ and $u(x) = \left(\ln \frac{1}{\|x\|}\right)^{1/4}$. Show that $u \in W^{1,2}(\Omega)$ but that it is not continuous.

In Class Exercises

25. Another "fundamental lemma" for L^1_{loc}-functions

Let $\Omega \subseteq \mathbb{R}^n$ be open and connected. Show that for $u \in L^1_{loc}(\Omega)$ if

$$\int_\Omega u(x)\nabla\phi(x)\, dx = 0 \text{ for all } \phi \in C_0^\infty(\Omega),$$

then u is constant on Ω. [Hint. Modify the proof of Proposition 3.23.]

26. An integration by parts.

Let $\Omega \subset \mathbb{R}^n$ be open and $u \in W_0^{1,2}(\Omega)$, $v \in W^{1,2}(\Omega)$. *Prove*

$$\int_\Omega u_{e_i} v_{e_j}\, \mathrm{d}\mu = \int_\Omega u_{e_j} v_{e_i}\, \mathrm{d}\mu$$

[Hint. Approximate u with functions from $C_0^\infty(\Omega)$.]