Martin Schmidt
Nicolas A. Hasse

Partial Differential Equations
Exercise sheet 5

14. März 2025

Preparatory Exercises

15. Second order differential operators Let $a_{ij}, \tilde{a}_{ij}, b_i, \tilde{b}_i, c, \tilde{c}_i$, and $\tilde{d}$ be real functions on the open set $\Omega \subset \mathbb{R}^n$. Any linear differential operator $L : C^2(\Omega) \to C(\Omega)$ of second order may be written as

$$(Lu)(x) = \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u(x) + \sum_{i=1}^{n} b_i(x)\partial_i u(x) + c(x)u(x). \tag{1}$$

This is called *general form* or *non-divergence form*. In contrast, we say that the operator is in *divergence form* when it is written as:

$$(Lu)(x) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\partial_i(\tilde{a}_{ij}(x)\partial_j u(x)) + \partial_i(\tilde{b}_i(x)u(x)) + \tilde{c}_i(x)\partial_i u\right) + \tilde{d}(x)u(x).$$

(a) Give the definition for a second order differential operator to be elliptic.

(b) Assume further that all the coefficient functions are differentiable. Show that the two forms are equivalent. Give the relationship between the coefficient functions.

(c) Define, for a constant symmetric matrix A, the second order differential operator L on $\mathbb{R}^n$.

$$(Lu)(x) := \nabla \cdot (A\nabla u(x))$$

Show that L is elliptic exactly when there is an invertible linear map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $L(u \circ \varphi) = (\triangle u) \circ \varphi$.
[Hint. A can be diagonalised by orthogonal matrices.]

Now let $\widetilde{\Omega} \subset \mathbb{R}^n$ be another open set and $\varphi : \Omega \to \widetilde{\Omega}$ a C^2-diffeomorphism. That is, φ is bijective, and both φ and φ^{-1} are twice continuously differentiable.

(d) Show that $\tilde{L}(\widetilde{u}) \circ \varphi = L(\widetilde{u} \circ \varphi)$ defines a second order differential operator $\widetilde{L}$ on $\widetilde{\Omega}$ for $\widetilde{u} \in C^2(\widetilde{\Omega})$. You may do this by writing $\widetilde{L}$ is general form.

(e) Now suppose that Ω und $\widetilde{\Omega}$ are bounded and that both functions φ, φ^{-1} and their derivatives extend continuously to the closure $\overline{\Omega}$, $\overline{\widetilde{\Omega}}$ respectively. Under this hypothesis, show that $\widetilde{L}$ is an elliptic operator exactly when L is. (Note, the relationship between L and $\widetilde{L}$ is symmetric, so it suffices to prove one direction only.)

In Class Exercises

16. Neumann Problems.

In this question we consider the Neumann problem for the Laplace equation on the unit ball in $\mathbb{R}^2$. [Note: One may freely use the Laplace-Operator in polar coordinates from Sheet 1.]

(a) Let $u \in C^2(\overline{B(0,1)})$ be a harmonic function on $B(0,1)$, with the polar coordinate form $u = u(r,\varphi)$ (for $0 \le r \le 1$ and $0 < \varphi \le 2\pi$). Show that

$$\int_{\partial B(0,1)} \frac{\partial u}{\partial r}(x)\mathrm{d}\sigma(x) = 0$$

holds.

(b) Hence show that there is no solution to the Neumann problem $\triangle u = 0$ on $B(0,1)$ with $\frac{\partial u}{\partial r} = \sin^2(\varphi)$ on $\partial B(0,1)$.

(c) Find all solutions to $\triangle u = 0$ on $B(0,1)$ with $\frac{\partial u}{\partial r} = \sin(\varphi)$ on $\partial B(0,1)$.

17. Compact Operators.

Let X, Y be Banach spaces. A linear, continuous mapping $T : X \to Y$ is called compact when for every bounded sequence $(x_m)_{m\in\mathbb{N}}$ in X there exists a subsequence $(x_{m_l})_{l\in\mathbb{N}}$ on which $(Tx_{m_l})_{l\in\mathbb{N}}$ converges.

(a) Show that a linear continuous mapping $T : X \to Y$ is compact exactly when the image of the unit ball $B(0,1) = \{x \in X \mid \|x\| < 1\}$ of X is relatively compact. (Recall that relatively compact means that the closure $\overline{T[B(0,1)]}$ is compact.)

(b) Let X be a Banach space and $\mathrm{id}_X : X \to X$ be the identity mapping. Show that id_X is a compact operator if and only if X is finite-dimensional.

18. A detail from the proof of Schauder's fixed point theorem.

Optional: Let K and $\tilde{K}$ be bounded, closed, convex subets of $\mathbb{R}^n$ with non-empty interiors. Prove that K and $\tilde{K}$ are homeomorphic.