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# Partial Differential Equations
## Exercise sheet 4

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**Preparatory Exercises**

**12. Spherical Means of Distributions.**

The purpose of this question is to provide some context into the definition of the weak mean value property and Weyl's lemma. We will essentially prove the *co-area formula.*

Let $\Psi : U \times [-T, T] \subset \mathbb{R}^n \to O \subset \mathbb{R}^n$ be a diffeomorphism. This is a smooth invertible function whose inverse function is also smooth. In particular, for each $t$ we know that $u \mapsto \Psi(u, t)$ is an $(n-1)$-dimensional submanifold. Denote these by $Y_t := \Psi[U \times \{t\}]$. Suppose further that $\partial_{u_i}\Psi \cdot \partial_t \Psi = 0$ for $i = 1, \ldots, n-1$ and $\|\partial_t \Psi\| = 1$.

**(a)** Check that spherical coordinates obey the assumptions on $\Psi$.

**(b)** Optional: Suppose we have vectors such that $b \cdot a_i = 0$ for $i = 1, \ldots, n-1$ and $\|b\| = 1$. Show that

$$|\det(a_1, \ldots, a_{n-1}, b)|^2 = |\det(a_1, \ldots, a_{n-1})^T (a_1, \ldots, a_{n-1})|.$$

Hint: Use the Gram matrix. Geometrically this is clear: the right hand side is the $n$-volume of a unit length right-prism and the left hand side is the $(n-1)$-volume of its cross-section.

**(c)** Argue that

$$\int_O f \, d\mu = \int_{U \times [-T,T]} f \circ \Psi \, |\det \Psi'| \, du \, dt = \int_{[-T,T]} \left( \int_{Y_t} f \, d\sigma \right) dt$$

**(d)** Consider the 'generalised mollifier' $\chi_\varepsilon : O \to \mathbb{R}$ defined by $\chi_\varepsilon(x) = \phi_\varepsilon(t(x))$ where $\phi_\varepsilon$ is a mollifier on $\mathbb{R}$. Complete the argument to show that

$$\lim_{\varepsilon \to 0} \int_O f \, \chi_\varepsilon \, d\mu = \int_{Y_0} f \, d\sigma.$$

This tempts us to define the integral of $F$ on $Y_0$ to be $\lim_{\varepsilon \to 0} F(\chi_\varepsilon)$ when this exists. This is similar to how we cannot always find the 'value' of a distribution at a point, but when we can it is $\lim_{\varepsilon \to 0} F(\phi_\varepsilon)$.

**(e)** Let us denote the functions $f$ and $g_x$ in the weak mean value property and Weyl's lemma by a common notation

$$g_{x,\psi}(y) := \frac{\psi(|x-y|)}{n\omega_n |x-y|^{n-1}}.$$

Use the above results to interpret the expression $U(g_{x,\psi})$. What is the significance of $\int \psi = 0$ compared with $\int_\psi = 1$?

**In Class Exercises**

**13. A detail in the proof of the Poisson Representation Formula (Poissonschen Darstellungsformel).**

We denote by $K(x, y)$ the Poisson kernel as in Section 2.3 of the lecture notes. This has the following properties (do *not* prove these properties again, refer to the lecture notes):

(i) $K(x,y) > 0$ for $x \in B(0,1),\ y \in \partial B(0,1)$.

(ii) $\int_{\partial B(0,1)} K(x,y) \mathrm{d}\sigma(y) = 1$ for $x \in B(0,1)$.

(iii) For all $x_0 \in \partial B(0,1)$, in the limit $x \to x_0$, $x \in B(0,1)$ the map $y \mapsto K(x,y)$ converges uniformly to 0 with respect to $y$ on compact subsets of $\partial B(0,1) \setminus \{x_0\}$.

Let a continuous function $u \in C(\partial B(0,1))$ be given. We define

$$\tilde{u} : B(0,1) \to \mathbb{R},\ x \mapsto \int_{\partial B(0,1)} K(x,y)u(y)\mathrm{d}\sigma(y)\ . \qquad (*)$$

Show that the function $\tilde{u}$ can be extended continuously to the boundary $\partial B(0,1)$ and that the extension on $\partial B(0,1)$ agrees with $u$.

[Hint: For any given $x_0 \in \partial B(0,1)$ consider $x \in B(0,1)$ in a neighbourhood of $x_0$ and break the integral $(*)$ into a piece close to $x_0$ and the "rest". Use the properties of $K$ given above to show that the "rest" is well behaved with respect to the limit and goes to zero. For the part close to $x_0$ use the continuity of $u$ to approximate the function values of $u(y)$ and $u(x_0)$.]

**14. A detail in the proof of the Weak Maximum Principle.**

Let $H$ be a real $n \times n$-matrix with

$$H = H^t \quad \text{and} \quad x^t H x \leq 0 \ \ \forall x.$$

We will show that there is a matrix $D$ such that $H = -D \cdot D^t$.

**(a)** Optional: Show that the eigenvalues of a real symmetric $n \times n$ matrix are real.

**(b)** Consider the map $f : \partial B(0,1) \to \mathbb{R}$ defined by $x \mapsto x^T H x$. Let $v$ be a maximum point of $f$. Show that $Hv = f(v)v$. [Hint. Consider a path $\alpha : (-\varepsilon, \varepsilon) \to \partial B(0,1)$ with $\alpha(0) = v$.]

**(c)** Suppose that $v$ is an eigenvector of $H$. Let $v^\perp := \{x \in \mathbb{R}^n \mid x \cdot v = 0\}$ be the orthogonal complement. Show that $Hv^\perp \subset v^\perp$.

**(d)** Prove inductively that there exists a matrix $O$ and real numbers $\lambda_i$ such that $H = O\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\, O^T$.

**(e)** Finally, show there is a matrix $D$ with $H = -D\, D^T$.