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Partial Differential Equations
Exercise sheet 3

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Preparatory Exercises

10. Spherical Means and Subharmonic functions.

(a) Show using the definition of integration on a submanifold, that

$$\int_{\partial B(a,r)} f(x)\, d\sigma(x) = r^{n-1} \int_{\partial B(0,1)} f(a+rz)\, d\sigma(z) \qquad \text{(3 Points)}$$

We introduce the follow general notation for spherical means

$$M(f,a,r) := \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(a,r)} f(x)\, d\sigma(x),$$

where ω_n is the volume of the unit ball. Show the following properties of the spherical mean.

(b) If c is a constant, $M(c,a,r) = c$. *(1 Point)*

(c) If $f \leq g$, then $M(f,a,r) \leq M(g,a,r)$, and $|M(f,a,r)| \leq M(|f|,a,r)$. *(1 Point)*

(d) If f is continuous at a, $\lim_{r\to 0^+} M(f,a,r) = f(a)$. *(2 Points)*

Let $\Omega \subset \mathbb{R}^n$ be an open connected domain. A twice continuously differentiable function $v : \overline{\Omega} \to \mathbb{R}$ is callled *subharmonic*, when $-\triangle v \leq 0$ on Ω.

(e) Let $v : \overline{\Omega} \to \mathbb{R}^n$ be subharmonic. Show for all $x \in \Omega$ and $r > 0$ with $B(x,r) \subset \Omega$ that $v(x) \leq M(v,x,r)$. [Hint: Adapt the proof of the mean value property] *(3 Points)*

(f) Prove the strong maximum principle for subharmonic functions: If v has a maximum on Ω then v constant. *(2 Points)*

In Class Exercises

11. Fundamental solution of the Laplace equation.

Let $n \geq 2$. In this question we investigate a function known as the fundamental solution of the Laplace equation:

$$\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R},\ x \mapsto \begin{cases} -\frac{1}{2\pi} \log(\|x\|) & \text{for } n = 2 \\ \frac{1}{n\,(n-2)\,\omega_n} \|x\|^{2-n} & \text{for } n \geq 3 \end{cases}$$

(a) Let $u \in C^2(\mathbb{R}^n \setminus \{0\})$ be rotationally symmetric. This means that $u(x) = v(\|x\|)$ for some twice continuously differentiable function $v : [0,\infty) \to \mathbb{R}$. Show that such solutions of the Laplace equation $\triangle u = 0$ have the form $A\Phi + B$ for constants $A, B \in \mathbb{R}$.
Hint: Use the Laplacian in n-dimensions in spherical coordinates (look it up). *(2 Points)*

(b) Calculate

$$\nabla\Phi = -\frac{1}{n\,\omega_n} \frac{x}{\|x\|^n}\ .$$

(2 Bonus Points)

(c) The fundamental solution Φ is chosen from this set of solutions to have two properties. The first is that it vanishes at infinity ($B = 0$). The second property ($A = 1$) is that

$$\int_{\partial B(0,r)} \nabla\Phi \cdot N \, d\sigma = -1$$

for all radii. Verify this. *(2 Points)*

(d) Can you see an easy proof that the integral in the previous question is independent of the radius? *(2 Bonus Points)*

Why is this function called the fundamental solution? Because in terms of distributions $-\triangle\Phi = \delta$ (using Φ also as the name of the corresponding distribution F_Φ). We will now prove this.

(e) Check that Φ is locally integrable, so that it does indeed define a distribution. *(1 Point)*

(f) For a test function ψ, why does $-\triangle\Phi(\psi) = -\Phi(\triangle\psi)$? *(1 Point)*

(g) Separate the integral $-\Phi(\triangle\psi)$ into one part that contains the singularity from Φ and another part that is singularity free:

$$I_\varepsilon := -\int_{B(0,\varepsilon)} \Phi\triangle\psi,$$
$$J_\varepsilon := -\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi\triangle\psi.$$

Show that $\lim_{\varepsilon\to 0+} I_\varepsilon = 0$. *(2 Points)*

(h) For every J_ε, prove that the following estimate holds:

$$J_\varepsilon = -\int_{\partial B(0,\varepsilon)} \psi \, \nabla\Phi \cdot N \mathrm{d}\sigma + L_\varepsilon \ ,$$

where L_ε is some expression that converges to zero as $\varepsilon \to 0$.
[Hint: Green's second formula.] *(3 Points)*

(i) Finally, prove $\lim_{\varepsilon\to 0} J_\varepsilon = \psi(0)$. *(2 Points)*

In total we showed that $-\triangle\Phi(\psi) = \psi(0) = \delta(\psi)$ for all test functions ψ. This shows that the negative Laplacian of Φ as a distribution is indeed the delta distribution.

What is the fundamental solution good for? It gives us a solution of the Poisson equation $-\triangle u = f$ in $\mathbb{R}^n$.

(j) If F is a distribution with compact support, why does $U = F * \Phi$ solve the Poisson equation $-\triangle U = F$ in the sense of distributions? *(1 Point)*

(k) More generally, if L is a linear differential operator, a distribution Φ is called a fundamental solution if $L\Phi = \delta$. Give a solution to the inhomogeneous equation $LU = F$. *(1 Bonus Point)*