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Partial Differential Equations
Exercise sheet 2

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Preparatory Exercises

6. Extension of Continuous Linear Operators. Let X be a normed vector space and $\bar{X}$ its completion. Let Y be a complete normed vector space. Suppose that $L : X \to Y$ is a continuous linear operator. This means that there is a constant C such that $\|Lx\| \leq C\|x\|$ for all $x \in X$. Show that there is a unique continuous linear operator $\bar{L} : \bar{X} \to Y$ extending L (ie $\bar{L}x = Lx$ for all $x \in X$). *(3 Points)*

7. On Convolutions.

(a) Let $f(x) = 1$ for $-1 \leq x \leq 1$ and 0 otherwise. Compute $f * f$. *(2 Points)*

(b) Show that the convolution of C_0^∞-functions on $\mathbb{R}^n$ is a bilinear, commutative, and associative operation. *(1+2 Points + 2 Bonus Points)*

(c) Denote a constant function on $\mathbb{R}$ by 1. The Heaviside function $H : \mathbb{R} \to \mathbb{R}$ is defined as $H(x) := 1$ for $x \geq 0$ and $H(x) := 0$ for $x < 0$. The derivative of the Dirac distribution δ' acts by $\delta'(\phi) = -\phi'(0)$. Let $\phi \in C_0^\infty(\mathbb{R})$ be a test function.

(i) Consider the distribution $\phi * P\delta'$. Which result from the script tells us that this distribution comes from a smooth function, even though δ' does not? *(1 Point)*

(ii) Prove that $\phi * P\delta' = F_{-\phi'}$. *(3 Points)*

(iii) Thereby show that $H * \delta' = \delta$ and $\delta' * 1 = 0$. *(2 Points)*

(iv) Complete the calculation of both $(H * \delta') * 1$ and $H * (\delta' * 1)$ in the sense of distributions and see that they are not equal. This shows that the convolution of distributions with non-compact support (on $\mathbb{R}$) is not necessarily associative, even when it is well-defined. *(1 Point)*

In Class Exercises

8. Distributions I.

(a) Show directly from Definition 2.6 that the Heaviside distribution

$$H : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto \int_0^\infty \phi(x)\,\mathrm{d}x$$

is a distribution on $\mathbb{R}$. *(2 Points)*

(b) By the definition of the derivative of a distribution

$$\partial H(\phi) = -H(\partial\phi) = -\int_0^\infty \phi'(x)\,\mathrm{d}x.$$

Simplify this expression in order to give a description of ∂H. (∂ here is the derivative in one-dimension. It seems weird to use an index.) *(3 Points)*

(c) What is the support of ∂H (in the sense of distributions)? Why does this show that there is no function $f \in L^1_{loc}(\mathbb{R})$ with $\partial H = F_f$? *(3 Points)*

(d) Consider a function $f \in C_0^\infty(\mathbb{R}^n)$. Show that $\partial_i(F_f) = F_{\partial_i f}$. What is the connection to Exercise 6?

(2 Points)

9. Distributions II.

(a) Show that

$$V(\phi) = \int_0^\infty \frac{\phi(x) - \phi(-x)}{x}\, dx$$

is a distribution on $\mathbb{R}$. Hint: Split the integral into $[0, 1]$ and $[1, \infty]$ and use the mean value theorem. *(2 Bonus Points)*

(b) What is the relation of V to x^{-1}? *(1 Bonus Point)*

(c) Show that the function $u : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \|x\|^k$ is a locally integrable function for $k > -n$. *(2 Bonus Points)*

(d) Let $n = 3$ and $k = -1$. Let $U = F_u$ be the distribution associated to u. It follows from 5(b) that $\partial_i u = -x_i\|x\|^{-3}$, which is also locally integrable, so expect $\partial_i U$ to correspond to $\partial_i u$. However we only know this correspondence holds in situations like Exercise 7(d). Using careful manipulation of the integrals (in particular, cut-out a ball $B(0, \varepsilon)$) show that our expectation holds. *(4 Bonus Points)*