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Partial Differential Equations
Exercise sheet 1

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1. Bumpy Road

Optional: Give an example of a function $u : \Omega \subset \mathbb{R} \to \mathbb{R}$ that is

(a) continuous but not differentiable.

(b) differentiable but not continuously differentiable.

(c) belongs to C^k but not C^{k+1}.

2. Vector Operators

Optional: Write in terms of components the formulas for the gradient ∇, the divergence $\nabla\cdot$, and the Laplacian $\triangle$.

3. The linear transport equation

Let $b \in \mathbb{R}^n$. The (homogeneous) linear transport equation with direction b is given by the following partial differential equation of first order:

$$\dot{u} + b \cdot \nabla u = 0\ . \tag{$*$}$$

This is a differential equation of $u = u(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, where $\dot{u}$ denotes the derivative of u with respect to $t \in \mathbb{R}$ and the gradient ∇u is taken with respect to $x \in \mathbb{R}^n$.

(a) Suppose that $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a C^1 solution of $(*)$. Show that u is constant on each of the parallel lines with direction $(b, 1) \in \mathbb{R}^n \times \mathbb{R}$. (Hint: Choose a line and parameterise it by s. Use the chain rule.) *(4 points)*

(b) Let $g \in C^1(\mathbb{R}^n)$. Prove that $u(x,t) := g(x - tb)$ is the *unique* solution of $(*)$ satisfying $u(\cdot, 0) = g$. *(5 points)*

4. In Colour.

Let Ω be a region in $\mathbb{R}^n$ and N the outer unit normal vector field on $\partial\Omega$. Let u, v be two C^2 real-valued functions on $\overline{\Omega}$.

(a) Show $v\triangle u = \nabla \cdot (v\nabla u) - \nabla u \cdot \nabla v$. *(2 points)*

(b) Prove the first Green formula

$$\int_\Omega v\triangle u\ dx = -\int_\Omega \nabla u \cdot \nabla v\ dx + \int_{\partial\Omega} v\nabla u \cdot N\ d\sigma.$$

(2 points)

(c) Using the first Green formula, prove the second Green formula

$$\int_\Omega (v\triangle u - u\triangle v)\ dx = \int_{\partial\Omega} (v\nabla u - u\nabla v) \cdot N\ d\sigma.$$

(1 points)

(d) Suppose further that v has support in Ω. This means that $\overline{\{x \in \Omega \mid v(x) \neq 0\}} \subsetneq \Omega$. Prove that

$$\int_\Omega v \triangle u \, dx = \int_\Omega u \triangle v \, dx$$

(1 points)

5. Laplacian and Laplace equation Laplace's equation is $\triangle u = 0$. A solution to Laplace's equation is called a harmonic function. We will discuss harmonic functions in further detail in the next chapter.

(a) Let $u, v : \Omega \to \mathbb{R}$ be harmonic functions. Show that the function $w(x) := u(x)v(x)$ is harmonic exactly when $\nabla u \perp \nabla v$. *(2 points)*

(b) Consider the function $u : \mathbb{R}^n \to \mathbb{R}, x \mapsto \|x\|$. Compute its gradient and Laplacian. *(3 points)*

(c) Optional: Let $u : \mathbb{R}^2 \to \mathbb{R}$ be twice-differentiable. Show for polar coordinates $x = r\cos(\varphi)$, $y = r\sin(\varphi)$ that

$$\triangle u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}.$$

(d) Optional: Let $u : \mathbb{R}^3 \to \mathbb{R}$ be twice differentiable.

(i) Show for cylindrical coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$ that

$$\triangle u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

(ii) Show for spherical coordinates $x = r\sin(\theta)\cos(\varphi)$, $y = r\sin(\theta)\sin(\varphi)$, $z = r\cos(\theta)$ that

$$\triangle u = \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial r}\left(r^2\sin(\theta)\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial u}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\sin(\theta)}\frac{\partial u}{\partial \varphi}\right)\right].$$

(e) Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain. Let $u \in C^2(\overline{\Omega})$ be a solution of the *boundary value problem*

$$\triangle u = 0 \quad \text{with} \quad u|_{\partial\Omega} = 0\,.$$

Show $u \equiv 0$. *(5 points)*

[Hint: Investigate $\int_\Omega u(\triangle u)\, \mathrm{d}x$ with the help of Green's first formula.]