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Partial Differential Equations
Exercise sheet 12

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41. Finite differences. Let $u \in W^{1,2}(B(0,1))$ be a weak solution of

$$L_0 u := \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u) = f \text{ in } B(0,1)$$

with $a_{ij} \in L^\infty(B(0,1))$ and $f \in L^2(B(0,1))$. Show that the finite difference

$$\partial_l^h u(x) := \frac{u(x+he_l) - u(x)}{h} \text{ for } x \in B(0, 1-|h|)$$

is a weak solution of

$$L_0 \partial_l^h u(x) = \partial_l^h f(x) - \sum_{i,j=1}^{n} \partial_i(\partial_l^h a_{ij} \partial_j u(x+he_l)),\ x \in B(0,1-|h|).$$

Solution. L_0 is an operator in divergence form. u is a weak solution of $L_0 u = f$ when

$$-\int_\Omega A\nabla u \cdot \nabla v = \langle f, v\rangle$$

for all $v \in W_0^{1,2}(\Omega)$. So we will compute the left hand side of the above for $\partial_l^h u$ and see if we can make it look like an element of $W_0^{1,2}(\Omega)^*$.

$$\begin{aligned}
-\int_\Omega A\nabla(\partial_l^h u)\cdot\nabla v &= \frac{1}{h}\int_\Omega \Big[A(x)\nabla u(x) - A(x)\nabla u(x+he_l)\Big]\cdot\nabla v(x) \\
&= \frac{1}{h}\int_\Omega \Big[A(x)\nabla u(x) - A(x+he_l)\nabla u(x+he_l) \\
&\qquad + A(x+he_l)\nabla u(x+he_l) - A(x)\nabla u(x+he_l)\Big]\cdot\nabla v(x) \\
&= \frac{1}{h}\int_\Omega \Big[-f(x) + f(x+he_l)\Big]v + \int_\Omega \partial_l^h A(x)\nabla u(x+he_l)\cdot\nabla v(x) \\
&= \langle \partial_l^h f, v\rangle + \int_\Omega \partial_l^h A(x)\nabla u(x+he_l)\cdot\nabla v(x) \\
&= \langle \partial_l^h f - \nabla\cdot[\partial_l^h A(x)\nabla u(x+he_l)], v\rangle.
\end{aligned}$$

This last expression is not to be taken literally, since we not know if the divergence actually exists, but in the sense of the pairing preceding Definition 4.1.

42. An Interpolation inequality.

Let $K = \overline{B(0,2)}$ and $(X, \|\cdot\|)$ be a Banach space that contains $C^1(K)$. In other words, there exists a continuous, injective linear map $I : C^1(K) \hookrightarrow X$. Examples are $X = L^2(K)$, which is similar to Theorem 4.11, and $X = C^0(K)$, which is similar to how how Lemma 3.44 (Interpolation of Sobolev spaces) is used in Theorem 4.34.

Show that a constant $C(n) < \infty$ exists such that

$$\|u\|_{C^2(K)} \le C(n)\left(\|D^2 u\|_{L^\infty(K)} + \|I(u)\|_X\right).$$

[Hint. Show that the embedding $C^2(K) \to C^1(K) \hookrightarrow X$ satisfies the assumptions of Ehrling's Lemma 3.3, ie that $T : C^2(K) \to C^1(K)$ is continuous and compact. Use the embedding theorems for space of continuous functions.]

Solution. Following the hint, we check that the assumptions of Ehrling's Lemma are satisfied. That I is continuous and injective is assumed in the question. The inclusion $T : C^2(K) \to C^1(K)$ can be factored as the inclusions $R : C^2(K) \to C^{1,1}(K)$ and $S : C^{1,1}(K) \to C^1(K)$. Proposition 3.15 shows that R is continuous and compact and Proposition 3.13 shows the same for S.

Now if we apply Ehrling's Lemma with $\varepsilon = 1/2$ we get

$$\|u\|_{C^1(K)} \leq 0.5\|u\|_{C^2(K)} + C'\|I(u)\|_X.$$

To get the inequality we desire, we apply the re-absorption trick

$$\begin{aligned}
\|u\|_{C^2(K)} &= \|u\|_{C^1(K)} + \|D^2u\|_\infty \leq 0.5\|u\|_{C^2(K)} + C'\|I(u)\|_X + \|D^2u\|_\infty \\
0.5\|u\|_{C^2(K)} &\leq C'\|I(u)\|_X + \|D^2u\|_\infty \\
\|u\|_{C^2(K)} &\leq 2C'\|I(u)\|_X + \|D^2u\|_\infty.
\end{aligned}$$

43. The interior Schauder Estimate.

At which place in the proof of the interior Schauder estimate 4.11 should we make a modification to instead prove the inequality

$$\|u\|_{C^{2,\alpha}(B(0,1))} \leq C(\Lambda, n, \alpha) \left(\|Lu\|_{C^{0,\alpha}(B(0,2))} + \|u\|_{L^1(B(0,2))}\right)$$

for $u \in C^{2,\alpha}(B(0,2))$? (Note in particular the last term on the right.)

Solution. As hinted at in the previous exercise, a sort of interpolation result is used in the proof of theorem 4.11. In the script we took $X = L^2(K)$ (top of page 75). If we want to end up with the L^1 norm in the estimate instead, clearly we should take $X = L^1(K)$. We should check that the inclusion $I : C^1(K) \to L^1(K)$ is a continuous injection. The injectivity is clear: if two C^1 functions are almost everywhere equal, they are equal. For continuity

$$\int_K |f| \leq \mu(K)\, \|f\|_\infty \leq \mu(K)\, \|f\|_{C^1}$$

does the job, since K is finite volume.