

27. Approximation by truncated Sobolev functions.

- (a) Let $\Omega \subset \mathbb{R}^n$ be open and $u, v \in W^{1,p}(\Omega)$. Show for $w(x) := \min\{u(x), v(x)\}$ that w also lies in $W^{1,p}(\Omega)$. Determine the weak derivatives of w .
 [Hint. Use the identities $\min\{a, b\} = \min\{a - b, 0\} + b$ and $\min\{a, 0\} = 0.5(a - |a|)$, and apply Propositions 3.29 (Chain Rule) and 3.30.]
- (b) Show using (a) that $\max\{u(x), v(x)\} \in W^{1,p}(\Omega)$ too.
- (c) Prove that when $u \in W^{1,p}(\Omega)$ then $w := \min\{u, 1\} \in W^{1,p}(\Omega)$. Calculate the first derivatives of w .
- (d) Show $L^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.
 [Hint. For $u \in W^{1,p}(\Omega)$ consider the sequence of truncations $u_n := \max\{-n, \min\{u, n\}\}$.]

Solution.

- (a) We can simplify if we write $\min\{u, v\} = \min\{u - v, 0\} + v$. Thus we need only consider the case $v = 0$. The function $f(y) = \min\{y, 0\}$ is a Lipschitz function with $f(0) = 0$, so the chain rule for Sobolev functions (Proposition 3.29) tells us that $w \in W^{1,p}$.

It remains to determine the derivative. The proof of Proposition 3.30 also works with the L^p norm. But even not assuming this, we know that the weak derivative of $\nabla|u| = \text{sign}(u)\nabla u$ and $\|\nabla|u|(x)\| \leq \|\nabla u(x)\|$ then shows that it is L^p . The derivative is therefore (not assuming $v = 0$)

$$\begin{aligned} \nabla w &= \nabla \left(v + \frac{1}{2}[u - v - |u - v|] \right) = \frac{1}{2} \nabla [u + v - |u - v|] \\ &= \frac{1}{2} [\nabla u + \nabla v - \text{sign}(u - v) (\nabla u - \nabla v)]. \end{aligned}$$

- (b) $\max\{u, v\} = -\min\{-u, -v\}$.
- (c) The difficulty here is that for an Ω with infinite area the constant functions are not L^p and we can not use part (a). But $|\min\{u(x), 1\}| \leq |u(x)|$ show us that $w \in L^p$ anyway. If we consider the function restricted to a bounded set $\Omega' \subset \Omega$ then we can apply part (a) and see that the weak derivative is given by

$$\nabla w = \chi_{\{u \geq 1\}} \nabla u.$$

Since $\|\nabla w(x)\| \leq \|\nabla u(x)\|$ this is also L^p .

- (d) As suggested by the hint, we consider $u_n := \max\{-n, \min\{u, n\}\}$. Since $\min\{u, n\} = n \min\{u/n, 1\}$ and $\max\{-n, v\} = -n \min\{-v/n, 1\}$ for $n > 0$, part (c) applies to u_n . It remains to show convergence in norms. But $f_n(x) := |u(x) - u_n(x)| = (|u(x)| - n)\chi_{|u(x)| > n}$ is a decreasing sequence of functions converging pointwise to zero and $f_0 = |u| \in L^p$, so by the dominated convergence theorem we get $\|u - u_n\|_p \rightarrow 0$. A similar result holds for the derivative with $\partial_i u_n = \partial_i u \chi_{|u| < n}$.

28. More Sobolev functions.

Let $p^{-1} + q^{-1} = 1$, $n > q$ and $\Omega = B(0, 1) \subset \mathbb{R}^n$. Choose $u \in C^1(\Omega \setminus \{0\})$ such that

$$\int_{\Omega \setminus \{0\}} |u(x)|^p d\mu < \infty \quad \text{and} \quad \int_{\Omega \setminus \{0\}} |\nabla u(x)|^p d\mu < \infty.$$

- (a) Choose any $\psi \in C^\infty(\mathbb{R})$ with $\psi(r) = 1$ for $r \geq 1$, $\psi(r) = 0$ for $r \leq \frac{1}{2}$, and $0 \leq \psi(r) \leq 1$. Let $\psi_k(x) = \psi(k|x|)$. Show that $\psi_k \rightarrow 1$ in $W^{1,q}(\Omega)$.
- (b) Define $u_k := u\psi_k$. Show that $\|\partial_i u - \partial_i u_k\|_1 \rightarrow 0$ as $k \rightarrow \infty$.
- (c) Complete the proof that $u \in W^{1,p}(\Omega)$ and $\partial_j u$ is its weak derivative.
- (d) Let $u : \Omega \setminus \{0\} \rightarrow \mathbb{R}$ be defined by $u(x) := \|x\|^\gamma$. Show $\partial^\alpha u(x) = P_\alpha(x)\|x\|^{\gamma-2|\alpha|}$, where P_α is a homogeneous degree $|\alpha|$ polynomial. The exact form of P_α is unimportant.
- (e) Using u from the previous part show that u belongs to $W^{k,p}(\Omega)$ for $\gamma > k - \frac{n}{p}$.

Solution.

- (a) These functions converge pointwise almost everywhere to 1 and are bounded, so the dominated convergence theorem shows they converge in L^q . We know that outside of $B(0, k^{-1})$ that ψ'_k is zero, so they converge pointwise to zero. However it remains to show the derivatives converge to zero in L^q . $\partial_i \psi_k = \psi'(k|x|)kx_i|x|^{-1}$, so

$$\begin{aligned} \|\partial_i \psi_k\|_q^q &= \int_{\Omega} |\psi'(k|x|)|^q k^q |x_i|^q |x|^{-q} = \int_{B(0, 1/k)} |\psi'(k|x|)|^q k^q |x_i|^q |x|^{-q} \\ &\leq \int_{B(0, 1/k)} \|\psi'\|_\infty^q k^q = \|\psi'\|_\infty^q k^q \times \omega_k k^{-n} \rightarrow 0 \end{aligned}$$

using the assumption that $n > q$.

- (b) ψ_k is identically zero on $B(0, (2k)^{-1})$ so $u_k := u\psi_k$ is $C^1(\Omega)$:

$$\begin{aligned} \|\partial_1 u - \partial_1 u\psi_k - u\partial_1 \psi_k\|_1 &\leq \|\partial_1 u - \partial_1 u\psi_k\|_1 + \|u\partial_1 \psi_k\|_1 \\ &\leq \|\partial_1 u\|_p \|1 - \psi_k\|_q + \|u\|_p \|\partial_1 \psi_k\|_q \rightarrow 0. \end{aligned}$$

- (c) If we try to directly compute the distributional derivative

$$\begin{aligned} - \int_{\Omega} u \partial_i \phi &= - \int_{B_\varepsilon} u \partial_i \phi - \int_{\Omega \setminus B_\varepsilon} u \partial_i \phi \\ &= - \int_{B_\varepsilon} u \partial_i \phi + \int_{\partial B_\varepsilon} u \phi \frac{x_i}{r} d\sigma + \int_{\Omega \setminus B_\varepsilon} \partial_i u \phi \end{aligned}$$

then we see that we have no good way to estimate the surface integral term. However

$$\left| \int_{\Omega} u_k \partial_i \phi - \int_{\Omega} u \partial_i \phi \right| \leq \|\partial_i \phi\|_\infty \|u\psi_k - u\|_1 \leq \|\partial_i \phi\|_\infty \|u\|_p \|\psi_k - 1\|_q \rightarrow 0$$

and

$$\left| \int_{\Omega} \partial_i u_k \phi - \int_{\Omega} \partial_i u \phi \right| \leq \|\phi\|_{\infty} \|\partial_i u_k - \partial_i u\|_1 \rightarrow 0$$

due to part (b). Together this shows

$$- \int_{\Omega} u \partial_i \phi = \lim - \int_{\Omega} u_k \partial_i \phi = \lim \int_{\Omega} \partial_i u_k \phi = \int_{\Omega} \partial_i u \phi,$$

i.e $\partial_i u$ is the weak derivative of u . By the given assumptions both are L^p .

(d) As a base case, we have $\partial^0 u = u = 1 \times |x|^{\gamma-0}$. Inductively

$$\begin{aligned} \partial_i P_{\alpha}(x) |x|^{\gamma-|\alpha|} &= P'_{\alpha}(x) |x|^{\gamma-|\alpha|} + (\gamma - |\alpha|) P_{\alpha}(x) |x|^{\gamma-|\alpha|-1} x_i |x|^{-1} \\ &= [P'_{\alpha}(x) |x|^2 + (\gamma - |\alpha|) P_{\alpha}(x) x_i] |x|^{\gamma-|\alpha|-2} \end{aligned}$$

and $P_{\alpha+e_i} := P'_{\alpha}(x) |x|^2 + (\gamma - |\alpha|) P_{\alpha}(x) x_i$ is indeed homogeneous of degree $|\alpha + e_i|$.

(e) For any homogeneous polynomial of degree $|\alpha|$ we can bound its growth crudely for all x using the triangle inequality

$$|P_{\alpha}(x)| = \left| \sum_{|\beta|=|\alpha|} C_{\beta} x^{\beta} \right| \leq \sum_{|\beta|=|\alpha|} |C_{\beta}| |x^{\beta}| \leq |x|^{|\alpha|} \sum_{|\beta|=|\alpha|} |C_{\beta}| = C |x|^{|\alpha|}.$$

(For non-homogeneous polynomial, we can only bound their growth for sufficiently large x .)

So for $\partial^{\alpha} u$ to belong to L^p it is sufficient for

$$\|\partial^{\alpha} u\|_p^p = \int_{\Omega} |P_{\alpha}|^p |x|^{p(\gamma-2|\alpha|)} \leq C \int_{\Omega} |x|^{p(\gamma-|\alpha|)} = C n \omega_n \int_0^1 r^{p(\gamma-|\alpha|)} r^{n-1} dr$$

to be finite, which holds when $p(\gamma - |\alpha|) + n - 1 > -1$. This is equivalent to $\gamma > |\alpha| - n/p$. If $\gamma > k - n/p$ then this holds for all $|\alpha| \leq k$ and all derivatives of u up and including order k belong to L^p .

29. An inequality for functions in $W_0^{2,2}(\Omega)$.

Let $\Omega \Subset \mathbb{R}^n$ be open and bounded, and $u \in W_0^{2,2}(\Omega)$. Prove the following inequality:

$$\|\nabla u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}^{1/2} \cdot \|\Delta u\|_{L^2(\Omega)}^{1/2}.$$

[Hint. Consider $u \in C_0^{\infty}(\Omega)$ and integrate $\int_{\Omega} |\nabla u|^2 d\mu$ by parts.]

Solution. For smooth functions note the identity $|\nabla u|^2 = \nabla \cdot (u \nabla u) - u \Delta u$. Let $u_k \in C_0^{\infty}(\Omega)$ converge to $u \in W_0^{2,2}(\Omega)$. Then

$$\begin{aligned} \|\nabla u\|_2^2 &= \int_{\Omega} |\nabla u|^2 = \lim \int_{\Omega} |\nabla u_k|^2 = \lim \int_{\Omega} \nabla \cdot (u_k \nabla u_k) - u_k \Delta u_k \\ &= \lim \int_{\partial\Omega} u_k \nabla u_k \cdot N d\sigma - \int_{\Omega} u_k \Delta u_k = \lim 0 - \int_{\Omega} u_k \Delta u_k \\ &\leq \lim \|u_k\|_2 \|\Delta u_k\|_2 = \|u\|_2 \|\Delta u\|_2. \end{aligned}$$

30. The Divergence theorem for Lipschitz continuous vector fields.

Let $\Omega \in \mathbb{R}^n$ be an open and bounded subset with boundary $\partial\Omega \in C^{0,1}$. We will show that the divergence theorem also holds for $f = (f_1, \dots, f_n) \in (C^{0,1}(\Omega))^n$:

$$\int_{\Omega} \nabla \cdot f \, d\mu = \int_{\partial\Omega} f \cdot N \, d\sigma. \quad (*)$$

Firstly we must modify Definition 1.7 appropriately. Concretely: We choose a finite open covering of coordinate charts $\{V_l\}_{l=1}^N$ and appropriate diffeomorphisms $\Phi_l : U_l \rightarrow V_l$, for open subsets $U_l \subset \mathbb{R}^{n-1}$. Next take a partition of unity $(h_l)_{l=1}^N$ and define

$$\int_{\partial\Omega} f \cdot N \, d\sigma = \sum_{l=1}^N \int_{U_l} h_l(f \cdot N) \circ \Phi_l \sqrt{\det(\Phi_l')^t \Phi_l'} \, d\mu. \quad (**)$$

- (a) *Show:* $\partial\Omega$ is continuously differentiable when, after a permutation of coordinates, Φ_l has the form $\Phi_l(y) = (y, \varphi_l(y))$, with $\varphi_l \in C^1(U_l, \mathbb{R})$.
- (b) *Show:* When $\partial\Omega$ is continuously differentiable and Φ_l has the form as in (a), then (**) becomes

$$\int_{\partial\Omega} f \cdot N \, d\sigma = \sum_{l=1}^N \int_{U_l} h_l f(y, \varphi_l(y)) \cdot (\nabla_y \varphi_l(y), -1) \, d^{n-1}y. \quad (***)$$

- (c) Let $A \in O(n, \mathbb{R})$ be an orthogonal matrix and f a smooth function.

Show: For $f_A = A \cdot f \circ A^{-1}$ the normal vector N_A of the transformed domain $\Omega_A = A[\Omega]$ satisfies the equation $N_A(x) = A \cdot N(A^{-1}x)$ and the divergence theorem (*) holds for (f_A, Ω_A) , if and only if it holds for (f, Ω) .

- (d) Let $\varphi \in C^{0,1}(B^{n-1}(0, \rho))$ with $\|\varphi\|_{\infty} < M$ and $f \in (W_0^{1,\infty}(B^{n-1}(0, \rho) \times (-M, M)))^n$. Then the following holds

$$\int_{B^{n-1}(0, \rho)} \int_{\varphi(y)}^M \nabla \cdot f(y, t) \, d^{n-1}y \, dt = \int_{B^{n-1}(0, \rho)} f(y, \varphi(y)) \cdot (\nabla_y \varphi, -1) \, d^{n-1}y.$$

[Hint: Approximationssatz 3.33]

- (e) *Show* that for $f = (f_1, \dots, f_n) \in (C^{0,1}(\Omega))^n$ the divergence theorem (*) hold.

[Hint: Show first that the expression in (c) holds also for $f \in (C^{0,1}(\Omega))^n$ and $\partial\Omega \in C^{0,1}$. Then use (d).]