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Partial Differential Equations
Exercise sheet 8

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27. Approximation by truncated Sobolev functions.

(a) Let $\Omega \subset \mathbb{R}^n$ be open and $u, v \in W^{1,p}(\Omega)$. Show for $w(x) := \min\{u(x), v(x)\}$ that w also lies in $W^{1,p}(\Omega)$. Determine the weak derivatives of w.
[Hint. Use the identities $\min\{a, b\} = \min\{a - b, 0\} + b$ and $\min\{a, 0\} = 0.5(a - |a|)$, and apply Propositions 3.29 (Chain Rule) and 3.30.]

(b) Show using (a) that $\max\{u(x), v(x)\} \in W^{1,p}(\Omega)$ too.

(c) Prove that when $u \in W^{1,p}(\Omega)$ then $w := \min\{u, 1\} \in W^{1,p}(\Omega)$. Calculate the first derivatives of w.

(d) Show $L^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.
[Hint. For $u \in W^{1,p}(\Omega)$ consider the sequence of truncations $u_n := \max\{-n, \min\{u, n\}\}$.]

28. More Sobolev functions.

Let $p^{-1} + q^{-1} = 1$, $n > q$ and $\Omega = B(0, 1) \subset \mathbb{R}^n$. Choose $u \in C^1(\Omega \backslash \{0\})$ such that

$$\int_{\Omega\backslash\{0\}} |u(x)|^p \, \mathrm{d}\mu < \infty \quad \text{and} \quad \int_{\Omega\backslash\{0\}} |\nabla u(x)|^p \, \mathrm{d}\mu < \infty.$$

(a) Choose any $\psi \in C^\infty(\mathbb{R})$ with $\psi(r) = 1$ for $r \geq 1$, $\psi(r) = 0$ for $r \leq \frac{1}{2}$, and $0 \leq \psi(r) \leq 1$. Let $\psi_k(x) = \psi(k|x|)$. Show that $\psi_k \to 1$ in $W^{1,q}(\Omega)$.

(b) Define $u_k := u\psi_k$. Show that $\|\partial_i u - \partial_i u_k\|_1 \to 0$ as $k \to \infty$.

(c) Complete the proof that $u \in W^{1,p}(\Omega)$ and $\partial_j u$ is its weak derivative.

(d) Let $u : \Omega \setminus \{0\} \to \mathbb{R}$ be defined by $u(x) := \|x\|^\gamma$. Show $\partial^\alpha u(x) = P_\alpha(x)\|x\|^{\gamma - 2|\alpha|}$, where P_α is a homogeneous degree $|\alpha|$ polynomial. The exact form of P_α is unimportant.

(e) Using u from the previous part show that u belongs to $W^{k,p}(\Omega)$ for $\gamma > k - \frac{n}{p}$.

29. An inequality for functions in $W_0^{2,2}(\Omega)$.

Let $\Omega \Subset \mathbb{R}^n$ be open and bounded, and $u \in W_0^{2,2}(\Omega)$. *Prove* the following inequality:

$$\|\nabla u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}^{1/2} \cdot \|\triangle u\|_{L^2(\Omega)}^{1/2}.$$

[Hint. Consider $u \in C_0^\infty(\Omega)$ and integrate $\int_\Omega |\nabla u|^2 \, \mathrm{d}\mu$ by parts.]

30. The Divergence theorem for Lipschitz continuous vector fields.

Let $\Omega \Subset \mathbb{R}^n$ be an open and bounded subset with boundary $\partial\Omega \in C^{0,1}$. We will show that the divergence theorem also holds for $f = (f_1, \ldots, f_n) \in (C^{0,1}(\Omega))^n$:

$$\int_\Omega \nabla \cdot f \, \mathrm{d}\mu = \int_{\partial\Omega} f \cdot N \, \mathrm{d}\sigma. \qquad (*)$$

Firstly we must modify Definition 1.7 appropriately. Concretely: We choose a finite open covering of coordinate charts $\{V_l\}_{l=1}^N$ and appropriate diffeomorphisms $\Phi_l : U_l \to V_l$, for open subsets $U_l \subset \mathbb{R}^{n-1}$. Next take a partition of unity $(h_l)_{l=1}^N$ and define

$$\int_{\partial\Omega} f \cdot N \mathrm{d}\sigma = \sum_{l=1}^N \int_{U_l} h_l (f \cdot N) \circ \Phi_l \sqrt{\det(\Phi_l')^t \Phi_l'} \mathrm{d}\mu. \tag{$**$}$$

(a) *Show*: $\partial\Omega$ is continuously differentiable when, after a permutation of coordinates, Φ_l has the form $\Phi_l(y) = (y, \varphi_l(y))$, with $\varphi_l \in C^1(U_l, \mathbb{R})$.

(b) *Show*: When $\partial\Omega$ is continuously differentiable and Φ_l has the form as in (a), then $(**)$ becomes

$$\int_{\partial\Omega} f \cdot N \, \mathrm{d}\sigma = \sum_{l=1}^N \int_{U_l} h_l f(y, \varphi_l(y)) \cdot (\nabla_y \varphi_l(y), -1) \, \mathrm{d}^{n-1} y. \tag{$***$}$$

(c) Let $A \in O(n, \mathbb{R})$ be an orthogonal matrix and f a smooth function.

Show: For $f_A = A \cdot f \circ A^{-1}$ the normal vector N_A of the transformed domain $\Omega_A = A[\Omega]$ satisfies the equation $N_A(x) = A \cdot N(A^{-1}x)$ and the divergence theorem $(*)$ holds for (f_A, Ω_A), if and only if it folds for (f, Ω).

(d) Let $\varphi \in C^{0,1}(B^{n-1}(0,\rho))$ with $\|\varphi\|_\infty < M$ and $f \in \left(W_0^{1,\infty}(B^{n-1}(0,\rho) \times (-M, M))\right)^n$. Then the following holds

$$\int_{B^{n-1}(0,\rho)} \int_{\varphi(y)}^M \nabla \cdot f(y,t) \, \mathrm{d}^{n-1} y \, dt = \int_{B^{n-1}(0,\rho)} f(y, \varphi(y)) \cdot (\nabla_y \varphi, -1) \, \mathrm{d}^{n-1} y.$$

[Hint: Approximationssatz 3.33]

(e) *Show* that for $f = (f_1, \ldots, f_n) \in (C^{0,1}(\Omega))^n$ the divergence theorem $(*)$ hold.

[Hint: Show first that the expression in (c) holds also for $f \in (C^{0,1}(\Omega))^n$ and $\partial\Omega \in C^{0,1}$. Then use (d).]