

19. Peano's existence theorem.

In this question we use Schauder's fixed point theorem to prove an existence theorem for ODEs. We will prove: Let $R = \{(x, w) \in \mathbb{R}^2 \mid |x| \leq a, |w| \leq b\}$ be a closed rectangle and $F : R \rightarrow \mathbb{R}$ a continuous function. Let c be the maximum of $|F|$. Then for $0 < h \leq \min\{a, b/c\}$ the following ODE has at least one solution $u : (-h, h) \rightarrow \mathbb{R}$

$$u' = F(x, u), \quad u(0) = 0.$$

- (a) In Schauder's theorem what conditions must X and G obey? Let $X = C([-h, h])$ and $G = \{u \in X \mid \|u\|_\infty \leq b\}$. Prove that they have the required conditions.
- (b) Consider $T : G \rightarrow X$ given by

$$(Tu)(x) = \int_0^x F(y, u(y)) dy.$$

Why is this a well defined operator on G ? Show that $T[G] \subseteq G$. Hence T is actually an operator $G \rightarrow G$.

- (c) Prove T is continuous. [Hint. F is uniformly continuous.]
- (d) Prove T is a compact operator.
[Hint. Arzela-Ascoli theorem: Consider a sequence of continuous functions $u_n : [-h, h] \rightarrow \mathbb{R}$. If this sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence that converges in X .]
- (e) Finish the proof of Peano's ODE existence theorem.

20. Properties of Hölder continuous functions.

Let $\Omega \subset \mathbb{R}^n$ be open.

- (a) Give the definitions for a function u to be α -Hölder continuous and to belong to $C^{0,\alpha}(\Omega)$.
- (b) Why is $\text{höl}_{\Omega,\alpha}$ not a norm?
- (c) Show a Hölder continuous function is uniformly continuous.
- (d) Suppose that $\alpha > 1$. Show that $u \in C^{0,\alpha}(\Omega)$ is differentiable and that $\nabla u \equiv 0$. This shows if Ω is connected and $\alpha > 1$ that $C^{0,\alpha}(\Omega)$ only contains the constant functions. For this reason we only consider $0 < \alpha \leq 1$.
- (e) Suppose that $u : [a, b] \rightarrow \mathbb{R}$ is continuously differentiable. Show that it is Hölder continuous for all $0 < \alpha \leq 1$.

21. Hölder-continuous functions on closed sets.

Optional: Let $\Omega \subset \mathbb{R}^n$ be an open subset of \mathbb{R}^n . These exercise considers the relationship between $C^{0,\alpha}(\Omega)$ and $C^{0,\alpha}(\overline{\Omega})$ (the latter is not defined in the script, but it has an obvious definition). Let $0 < \alpha \leq 1$ and $u \in C^{0,\alpha}(\Omega)$.

- (a) Give a function $f : \bar{\Omega} \rightarrow \mathbb{R}$ that belongs to $C(\Omega)$ but not $C(\bar{\Omega})$, either for general Ω or a particular choice.
- (b) Show that there is a unique function $\tilde{u} \in C(\bar{\Omega})$ with $\tilde{u}|_{\Omega} = u$. [Hint. Use uniform continuity.]
- (c) Prove that $\text{höl}_{\bar{\Omega},\alpha} \tilde{u} = \text{höl}_{\Omega,\alpha} u$.
- (d) What can you then say about the relationship between $C^{0,\alpha}(\Omega)$ and $C^{0,\alpha}(\bar{\Omega})$?

22. Examples of Hölder continuous functions.

- (a) For $0 < b \leq 1$ define $f_b : (0, 1) \rightarrow \mathbb{R}$ by $x \mapsto x^b$. To which Hölder spaces does f_b belong? Compute its Hölder constants höl_{α} .
[Hint. Consider the function $h(z) = (1 - z^b)(1 - z)^{-\alpha}$.]
- (b) Now define $g_b : (0, \infty) \rightarrow \mathbb{R}$ by $x \mapsto x^b$. To which Hölder spaces does g_b belong? Compute its Hölder constants höl_{α} .
- (c) Define $h : [0, 0.5] \rightarrow \mathbb{R}$ by $h(0) = 0$ and $h(x) = (\ln x)^{-1}$ otherwise. Show that this function is continuous but not Hölder continuous. Can you explain why?
- (d) Explain parts (a) and (b) with respect to Proposition 3.13.