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Partial Differential Equations
Exercise sheet 5

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15. Second order differential operators Let $a_{ij}, \tilde{a}_{ij}, b_i, \tilde{b}_i, c, \tilde{c}_i$, and $\tilde{d}$ be real functions on the open set $\Omega \subset \mathbb{R}^n$. Any linear differential operator $L : C^2(\Omega) \to C(\Omega)$ of second order may be written as

$$(Lu)(x) = \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u(x) + \sum_{i=1}^{n} b_i(x)\partial_i u(x) + c(x)u(x). \tag{1}$$

This is called *general form* or *non-divergence form.* In contrast, we say that the operator is in *divergence form* when it is written as:

$$(Lu)(x) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\partial_i(\tilde{a}_{ij}(x)\partial_j u(x)) + \partial_i(\tilde{b}_i(x)u(x)) + \tilde{c}_i(x)\partial_i u\right) + \tilde{d}(x)u(x).$$

(a) Give the definition for a second order differential operator to be elliptic.

(b) Assume further that all the coefficient functions are differentiable. Show that the two forms are equivalent. Give the relationship between the coefficient functions.

(c) Define, for a constant symmetric matrix A, the second order differential operator L on $\mathbb{R}^n$.

$$(Lu)(x) := \nabla \cdot (A\nabla u(x))$$

Show that L is elliptic exactly when there is an invertible linear map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $L(u \circ \varphi) = (\triangle u) \circ \varphi$.
[Hint. A can be diagonalised by orthogonal matrices.]

Now let $\widetilde{\Omega} \subset \mathbb{R}^n$ be another open set and $\varphi : \Omega \to \widetilde{\Omega}$ a C^2-diffeomorphism. That is, φ is bijective, and both φ and φ^{-1} are twice continuously differentiable.

(d) Show that $\tilde{L}(\widetilde{u}) \circ \varphi = L(\widetilde{u} \circ \varphi)$ defines a second order differential operator $\widetilde{L}$ on $\widetilde{\Omega}$ for $\widetilde{u} \in C^2(\widetilde{\Omega})$. You may do this by writing $\widetilde{L}$ is general form.

(e) Now suppose that Ω und $\widetilde{\Omega}$ are bounded and that both functions φ, φ^{-1} and their derivatives extend continuously to the closure $\overline{\Omega}$, $\overline{\widetilde{\Omega}}$ respectively. Under this hypothesis, show that $\widetilde{L}$ is an elliptic operator exactly when L is. (Note, the relationship between L and $\widetilde{L}$ is symmetric, so it suffices to prove one direction only.)

Solution.

(a) This is Definition 2.11. The coefficient functions should be bounded and the (uniform) ellipticity condition should hold: there is a positive constant Λ^{-1} such that for all points $x \in \Omega$ and vectors $\lambda \in \mathbb{R}^n$ we have $\lambda^T A(x)\lambda \geq \Lambda^{-1}|\lambda|$, where $A = (a_{ij})$ is the matrix of coefficients.

Both sides of the ellipticity are quadratic in λ, so it is possible to rephrase the condition to only use unit vectors: $\lambda^T A(x)\lambda \geq \Lambda^{-1}$ for all unit vectors $\lambda \in \mathbb{R}^n$ and all $x \in \Omega$. This condition is stronger than simply being positive at every point since the domain Ω is open and so if we only assume that it is positive we could have $\inf_{x\in\Omega} \lambda^T A(x)\lambda = 0$.

(b) Since the coefficients are differentiable we can expand the divergence form (I'll leave off the function (x)'s)

$$\begin{aligned}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\partial_i(\tilde a_{ij}\partial_j u)+\partial_i(\tilde b_i u)+\tilde c_i\partial_i u\right)+\tilde d u\\ =\sum_{i,j=1}^{n}\left(\partial_i\tilde a_{ij}\partial_j u+\tilde a_{ij}\partial_i\partial_j u\right)+\sum_{i=1}^{n}\left(\partial_i\tilde b_i u+\tilde b_i\partial_i u+\tilde c_i\partial_i u\right)+\tilde d u\\ =\sum_{i,j=1}^{n}\tilde a_{ij}\partial_i\partial_j u+\sum_{i=1}^{n}\left(\tilde b_i+\tilde c_i+\sum_{j=1}^{n}\partial_j\tilde a_{ji}\right)\partial_i u+\left(\tilde d+\sum_{i=1}^{n}\partial_i\tilde b_i\right)u\end{aligned}$$

This shows that operators in divergence form can be written in non-divergence form. For the converse we need to solve

$$\tilde a_{ij}=a_{ij},\qquad \tilde b_i+\tilde c_i+\sum_{j=1}^{n}\partial_j\tilde a_{ji}=b_i,\qquad \tilde d+\sum_{i=1}^{n}\partial_i\tilde b_i=c$$

for the tilde-coefficients. The first equation tells us $\tilde a_{ij}$. The second equation allows us some choice. We can take, for example, $\tilde b_i=0$ and then

$$\tilde c_i=b_i-\sum_{j=1}^{n}\partial_j a_{ji}.$$

Under this assumption, $\tilde d=c$.

Later in the course we will see that, generally speaking, if we can make $\tilde c_i=0$ then we can get stronger results. But to convert a non-divergence form operator into a divergence form operator with $\tilde c_i=0$ requires (more-or-less) that a_{ij} be twice differentiable and b_i be once differentiable.

(c) In coordinates $Lu=\sum_{i,j}\partial_i(A_{ij}\partial_j u)=\sum_{i,j}A_{ij}\partial_i\partial_j u$. Incidently, we see why the an operator in divergence form has that name. We know that $A=O^TDO$ for a diagonal matrix D and an orthogonal matrix O. In coordinates $A_{ij}=\sum_{k,l}O_{ki}D_{kl}O_{lj}$. Consider the coordinate change $y=Ox$, which is the same as $x=O^Ty$. By the chain rule $\partial_{y_l}=\sum_j\partial_{y_l}x_j\partial_{x_j}=\sum_j O_{lj}\partial_{x_j}$. Using this, we can write

$$Lu=\sum_{i,j,k,l}O_{ki}D_{kl}O_{lj}\partial_{x_i}\partial_{x_j}u=\sum_{k,l}D_{kl}\partial_{y_k}\partial_{y_l}u=\sum_k D_{kk}\partial_{y_k}^2 u,$$

since D is diagonal.

Now let us consider the ellipticity condition. For any vector $\lambda\in\mathbb{R}^n$

$$\lambda^TA\lambda=\lambda^TO^TDO\lambda=(O\lambda)^TD(O\lambda)$$

and $|O\lambda|=|\lambda|$, so A fulfils the ellipticity condition if and only if D does. By considering the standard basis vectors $(1,0,\dots)$, etc, we see that D fulfils the ellipticity condition if and only if its diagonal entries are positive. If we rescale the coordinates by the square root of the diagonal entries, $z_i=D_{ii}^{-0.5}y_i$ we get that $Lu=\sum\partial_{z_i}^2u=\triangle_z u$.

(d) The working in this part is similar to in the previous part. In the previous part we calculated with the specific form of the coordinate changes we needed. In this part we use more general formulas. Of course you could use the formula in this part and apply them to the previous part with $\varphi(x) = D^{-0.5}Ox$.

By the chain rule

$$\begin{aligned}\partial_i(\tilde{u} \circ \varphi)(x) &= \sum_k \partial_k \tilde{u}(\varphi(x))\partial_i \varphi_k(x) \\ \partial_j \partial_i(\tilde{u} \circ \varphi)(x) &= \sum_k \partial_j[\partial_k \tilde{u}(\varphi(x))]\partial_i \varphi_k(x) + \partial_k \tilde{u}(\tilde{x})\partial_j \partial_i \varphi_k(x) \\ &= \sum_{k,l} \partial_l \partial_k \tilde{u}(\varphi(x))\partial_j \varphi_l(x)\partial_i \varphi_k(x) + \sum_k \partial_k \tilde{u}(\varphi(x))\partial_j \partial_i \varphi_k(x).\end{aligned}$$

If we substitute this into a formula of L in general form and regroup the terms, then we see that $\tilde{L}$ is also in general form.

(e) From the previous part we see that the highest order terms $\tilde{a}_{ij}$ of $\tilde{L}$ are

$$\tilde{a}_{ij} = \sum_{i,j} a_{ij} \partial_j \varphi_l \partial_i \varphi_k.$$

If we write this in terms of matrices with $J_{jl} = \partial_j \varphi_l$ the derivative of φ then $\tilde{a} = J^T a J$. Moreover, we know that J is always invertible because it is a diffeomorphism. Thus $|J\lambda| > 0$ and by a similar argument to part (c) we know that $\lambda^T \tilde{a} \lambda > 0$ for all points x and unit length vectors λ.

To strengthen this argument and show that $\inf \lambda^T \tilde{a} \lambda > 0$ we need to use the additional hypotheses. Since we know that all the entries of J and J^{-1} extend to the boundary continuously, by taking the limit of $J(x)J^{-1}(x) = I$ we see that J is also invertible on the boundary. Thus $(x, \lambda) \mapsto |J(x)\lambda|$ is a continuous positive function on the compact set $\overline{\Omega} \times \partial B(0,1)$ must have a positive minimum. This shows finally that

$$\inf_{(x,\lambda) \in \Omega \times \partial B(0,1)} \lambda^T \tilde{a} \lambda = \inf_{(x,\lambda) \in \Omega \times \partial B(0,1)} (J\lambda)^T a (J\lambda) \geq \inf_{(x,\lambda) \in \Omega \times \partial B(0,1)} \Lambda^{-1} |J\lambda|^2 > 0.$$

16. Neumann Problems.

In this question we consider the Neumann problem for the Laplace equation on the unit ball in $\mathbb{R}^2$. [Note: One may freely use the Laplace-Operator in polar coordinates from Sheet 1.]

(a) Let $u \in C^2(\overline{B(0,1)})$ be a harmonic function on $B(0,1)$, with the polar coordinate form $u = u(r, \varphi)$ (for $0 \leq r \leq 1$ and $0 < \varphi \leq 2\pi$). Show that

$$\int_{\partial B(0,1)} \frac{\partial u}{\partial r}(x) \mathrm{d}\sigma(x) = 0$$

holds.

(b) Hence show that there is no solution to the Neumann problem $\triangle u = 0$ on $B(0,1)$ with $\frac{\partial u}{\partial r} = \sin^2(\varphi)$ on $\partial B(0,1)$.

(c) Find all solutions to $\triangle u = 0$ on $B(0,1)$ with $\frac{\partial u}{\partial r} = \sin(\varphi)$ on $\partial B(0,1)$.

Solution.

(a) Note that in polar coordinates the outward pointing normal of the ball is simply $(1,0)$ because it is radial and length 1. Therefore we see that the integral is already in the form of the divergence theorem:

$$\int_{\partial B} \frac{\partial u}{\partial r} \, \mathrm{d}\sigma = \int_{\partial B} \nabla u \cdot N \, \mathrm{d}\sigma = \int_B \Delta u \, \mathrm{d}\sigma = 0.$$

(b) One can compute the integral in part (a) exactly, but the following estimate is sufficent. For $\pi/4 \leq \varphi \leq 3\pi/4$ we have $\sin^2 \varphi \geq 0.5$. Therefore

$$\int_{\partial B(0,1)} \frac{\partial u}{\partial r}(x) \, \mathrm{d}\sigma(x) \geq \int_{\pi/4}^{3\pi/4} \sin^2 \varphi \, \mathrm{d}\varphi \geq \pi/4.$$

There can be no solution in this case because it would not obey the property established in part (a).

(c) For this question we guess that the solution is of the form $u(r,\varphi) = f(r)\sin\varphi$. Laplace's equation then reads

$$\left(f'' + \frac{1}{r}f'\right)\sin\varphi - \frac{1}{r^2}f\sin\varphi = 0 \Rightarrow r^2 f'' + rf' - f = 0 \Rightarrow f(r) = Ar^{-1} + Br.$$

Since we want the solution to be defined on the disc, we must choose $A = 0$. The boundary condition $\partial_r u|_{\partial B} = \sin\varphi$ then requires that $f'(1) = B = 1$. Hence the solution is $u(r,\varphi) = r\sin\varphi$. Writing this in Cartesian coordinates make it trivial to see why this is harmonic: $u(x,y) = y$. We can add any constant to this without changing its Laplacian or normal derivative. Theorem 2.18 tells us that these are all the possible solutions.

17. Compact Operators.

Let X, Y be Banach spaces. A linear, continuous mapping $T : X \to Y$ is called compact when for every bounded sequence $(x_m)_{m\in\mathbb{N}}$ in X there exists a subsequence $(x_{m_l})_{l\in\mathbb{N}}$ on which $(Tx_{m_l})_{l\in\mathbb{N}}$ converges.

(a) Show that a linear continuous mapping $T : X \to Y$ is compact exactly when the image of the unit ball $B(0,1) = \{x \in X \mid \|x\| < 1\}$ of X is relatively compact. (Recall that relatively compact means that the closure $\overline{T[B(0,1)]}$ is compact.)

(b) Let X be a Banach space and $\mathrm{id}_X : X \to X$ be the identity mapping. Show that id_X is a compact operator if and only if X is finite-dimensional.

Solution.

(a) Suppose $\overline{T[B(0,1)]}$ is compact. Let (x_n) be a bounded sequence: ie there exists and R with $\|x_n\| < R$. Then $x'_n := R^{-1}x_n \in B(0,1)$ and so (Tx'_n) must have a convergent subsequence (Tx'_{n_k}). But then $Tx_{n_k} = RTx'_{n_k}$ is also convergent.

Conversely suppose that T is compact. Take any sequence $y_n \in \overline{T[B(0,1)]}$. There are elements $y_{n,k} \in T[B(0,1)]$ with $y_{n,k} \to y_n$ as $k \to \infty$. Let $x_{n,k} \in B(0,1)$ be elements with $Tx_{n,k} = y_{n,k}$. All these vectors $\|x_{n,k}\| < 1$ so the sequence $(x_{n,n})$ is bounded. Because T is compact, that means that $y_{n,n}$ has a convergent subsequence $y_{n_l,n_l} \to y \in \overline{T[B(0,1)]}$. Finally

$$\lim_{l\to\infty} \|y - y_{n_l}\| \leq \lim_{l\to\infty} \|y - y_{n_l,n_l}\| + \lim_{l\to\infty} \|y_{n_l,n_l} - y_{n_l}\| = 0 + 0$$

shows that (y_{n_l}) is a convergent subsequence of (y_n).

(b) The closed unit ball of a Banach space X is compact if and only if X is finite dimensional. Hence id_X is compact iff $\overline{\mathrm{id}_X[B(0,1)]}$ is compact iff $\overline{B(0,1)}$ is compact iff X is finite dimensional.

18. A detail from the proof of Schauder's fixed point theorem.

Optional: Let K and $\tilde{K}$ be bounded, closed, convex subets of $\mathbb{R}^n$ with non-empty interiors. Prove that K and $\tilde{K}$ are homeomorphic.

Solution. Because homeomorphic is a transitive relation, it is enough to prove this for $\tilde{K} = \overline{B(0,1)}$.

We can assume that $0 \in \operatorname{int} K$. Let $v \in \partial B(0,1)$. By the properties of K we know that $\{t \in \mathbb{R}_{\geq 0} \mid tv \in K\}$ is an interval of the form $[0,d]$. Or put differently $\mathbb{R}_{\geq 0}v \cap K = [0,d]v$. The idea is to rescale these rays by d to make K into a sphere.

Let us consider the 'furthest distance' function. Define $d : \partial B(0,1) \to \mathbb{R}_{>0}$ by $d(v) = \max\{t \in \mathbb{R}_{>0} \mid tv \in K\}$. Because K is bounded, d must have an upper bound R. We know that this function is strictly positive because 0 is in the interior of K. Moreover, d must be bounded from below by a positive constant r because otherwise we would have a sequence v_k with $d(v_k) \to 0$. But then the elements $d(v_k)v_k \in \partial K$ converge to 0, which contradicts the fact that 0 is in the interior of K.

We now show that d is continuous. Suppose that d were not continuous. That means there is a sequence $v_n \to v$ in $\partial B(0,1)$ such that $|d(v_k) - d(v)| > C$ for some positive constant C. On the other hand, consider $d(v_k)v_k \in \partial K$. Since ∂K is compact, there is a subsequence converging to $x \in \partial K$. We know that $x \neq 0$ so write $x = d(\hat{x})\hat{x}$. By normalising (a continuous function), we see that $\hat{x} = v$. For this subsequence we have

$$\begin{aligned}\|d(v_k)v_k - d(v)v\| &= \|d(v_k)v_k - d(v_k)v + d(v_k)v - d(v)v\| \\ &\geq \Big| d(v_k)\,\|v_k - v\| - |d(v_k) - d(v)| \Big|\end{aligned}$$

Because d is bounded, we know that $d(v_k)\,\|v_k - v\|$ converges to 0. For large k therefore this inequality cannot hold since $|d(v_k) - d(v)| > C$. This is a contradiction.

Now we can define the homeomorphism $\varphi : K \to \overline{B(0,1)}$ by $\varphi(0) = 0$ and $\varphi(x) = d(\hat{x})^{-1}x$. Since $x \mapsto \hat{x}$ is continuous away from $x = 0$ and d is strictly positive, φ is continuous away from $x = 0$. If we have a sequence $x_k \to 0$ then $\|\varphi(x_k)\| \leq r^{-1}\|x_k\|$ shows that $\varphi(x_k) \to 0$. Hence φ is continuous. It has an inverse $\varphi^{-1}(0) = 0$ and $\varphi^{-1}(x) = d(\hat{x})x$, which is also continuous by essentially the same argument.