

**15. Second order differential operators** Let  $a_{ij}, \tilde{a}_{ij}, b_i, \tilde{b}_i, c, \tilde{c}_i$ , and  $\tilde{d}$  be real functions on the open set  $\Omega \subset \mathbb{R}^n$ . Any linear differential operator  $L : C^2(\Omega) \rightarrow C(\Omega)$  of second order may be written as

$$(Lu)(x) = \sum_{i,j=1}^n a_{ij}(x) \partial_i \partial_j u(x) + \sum_{i=1}^n b_i(x) \partial_i u(x) + c(x)u(x). \quad (1)$$

This is called *general form* or *non-divergence form*. In contrast, we say that the operator is in *divergence form* when it is written as:

$$(Lu)(x) = \sum_{i=1}^n \left( \sum_{j=1}^n \partial_i (\tilde{a}_{ij}(x) \partial_j u(x)) + \partial_i (\tilde{b}_i(x) u(x)) + \tilde{c}_i(x) \partial_i u \right) + \tilde{d}(x)u(x).$$

- (a) Give the definition for a second order differential operator to be elliptic.
- (b) Assume further that all the coefficient functions are differentiable. Show that the two forms are equivalent. Give the relationship between the coefficient functions.
- (c) Define, for a constant symmetric matrix  $A$ , the second order differential operator  $L$  on  $\mathbb{R}^n$ .

$$(Lu)(x) := \nabla \cdot (A \nabla u(x))$$

Show that  $L$  is elliptic exactly when there is an invertible linear map  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $L(u \circ \varphi) = (\Delta u) \circ \varphi$ .

[Hint.  $A$  can be diagonalised by orthogonal matrices.]

Now let  $\tilde{\Omega} \subset \mathbb{R}^n$  be another open set and  $\varphi : \Omega \rightarrow \tilde{\Omega}$  a  $C^2$ -diffeomorphism. That is,  $\varphi$  is bijective, and both  $\varphi$  and  $\varphi^{-1}$  are twice continuously differentiable.

- (d) Show that  $\tilde{L}(\tilde{u}) \circ \varphi = L(\tilde{u} \circ \varphi)$  defines a second order differential operator  $\tilde{L}$  on  $\tilde{\Omega}$  for  $\tilde{u} \in C^2(\tilde{\Omega})$ . You may do this by writing  $\tilde{L}$  is general form.
- (e) Now suppose that  $\Omega$  und  $\tilde{\Omega}$  are bounded and that both functions  $\varphi, \varphi^{-1}$  and their derivatives extend continuously to the closure  $\bar{\Omega}, \bar{\tilde{\Omega}}$  respectively. Under this hypothesis, show that  $\tilde{L}$  is an elliptic operator exactly when  $L$  is. (Note, the relationship between  $L$  and  $\tilde{L}$  is symmetric, so it suffices to prove one direction only.)

**16. Neumann Problems.**

In this question we consider the Neumann problem for the Laplace equation on the unit ball in  $\mathbb{R}^2$ . [Note: One may freely use the Laplace-Operator in polar coordinates from Sheet 1.]

- (a) Let  $u \in C^2(\overline{B(0,1)})$  be a harmonic function on  $B(0,1)$ , with the polar coordinate form  $u = u(r, \varphi)$  (for  $0 \leq r \leq 1$  and  $0 < \varphi \leq 2\pi$ ). Show that

$$\int_{\partial B(0,1)} \frac{\partial u}{\partial r}(x) d\sigma(x) = 0$$

holds.

- (b) Hence show that there is no solution to the Neumann problem  $\Delta u = 0$  on  $B(0, 1)$  with  $\frac{\partial u}{\partial r} = \sin^2(\varphi)$  on  $\partial B(0, 1)$ .
- (c) Find all solutions to  $\Delta u = 0$  on  $B(0, 1)$  with  $\frac{\partial u}{\partial r} = \sin(\varphi)$  on  $\partial B(0, 1)$ .

### 17. Compact Operators.

Let  $X, Y$  be Banach spaces. A linear, continuous mapping  $T : X \rightarrow Y$  is called compact when for every bounded sequence  $(x_m)_{m \in \mathbb{N}}$  in  $X$  there exists a subsequence  $(x_{m_l})_{l \in \mathbb{N}}$  on which  $(Tx_{m_l})_{l \in \mathbb{N}}$  converges.

- (a) Show that a linear continuous mapping  $T : X \rightarrow Y$  is compact exactly when the image of the unit ball  $B(0, 1) = \{x \in X \mid \|x\| < 1\}$  of  $X$  is relatively compact. (Recall that relatively compact means that the closure  $\overline{T[B(0, 1)]}$  is compact.)
- (b) Let  $X$  be a Banach space and  $\text{id}_X : X \rightarrow X$  be the identity mapping. Show that  $\text{id}_X$  is a compact operator if and only if  $X$  is finite-dimensional.

### 18. A detail from the proof of Schauder's fixed point theorem.

Optional: Let  $K$  and  $\tilde{K}$  be bounded, closed, convex subsets of  $\mathbb{R}^n$  with non-empty interiors. Prove that  $K$  and  $\tilde{K}$  are homeomorphic.