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Partial Differential Equations
Exercise sheet 3

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10. Spherical Means and Subharmonic functions.

(a) Show using the definition of integration on a submanifold, that

$$\int_{\partial B(a,r)} f(x)\, d\sigma(x) = r^{n-1} \int_{\partial B(0,1)} f(a+rz)\, d\sigma(z) \qquad \text{(3 Points)}$$

We introduce the follow general notation for spherical means

$$M(f,a,r) := \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(a,r)} f(x)\, d\sigma(x),$$

where ω_n is the volume of the unit ball. Show the following properties of the spherical mean.

(b) If c is a constant, $M(c,a,r) = c$. *(1 Point)*

(c) If $f \leq g$, then $M(f,a,r) \leq M(g,a,r)$, and $|M(f,a,r)| \leq M(|f|,a,r)$. *(1 Point)*

(d) If f is continuous at a, $\lim_{r\to 0^+} M(f,a,r) = f(a)$. *(2 Points)*

Let $\Omega \subset \mathbb{R}^n$ be an open connected domain. A twice continuously differentiable function $v : \overline{\Omega} \to \mathbb{R}$ is callled *subharmonic*, when $-\triangle v \leq 0$ on Ω.

(e) Let $v : \overline{\Omega} \to \mathbb{R}^n$ be subharmonic. Show for all $x \in \Omega$ and $r > 0$ with $B(x,r) \subset \Omega$ that $v(x) \leq M(v,x,r)$. [Hint: Adapt the proof of the mean value property] *(3 Points)*

(f) Prove the strong maximum principle for subharmonic functions: If v has a maximum on Ω then v constant. *(2 Points)*

Solution.

(a) The sphere here is a submanifold of dimension $n-1$ and a compact set, so $K = A = \partial B(a,r)$ in terms of the definition. Cover the sphere by open sets O_i that can be parameterised $\Phi_i : U_i \subset \mathbb{R}^{n-1} \to O_i \cap A$ and let h_l be the corresponding partition of unity. The thing to recognise then is that if we translate and rescale these things, we also get a complete parameterisation of the unit sphere. Let $L : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto rx + a$. Then $\tilde{\Phi} := L^{-1} \circ \Phi_i : U_i \to L^{-1}[O_i]$ is a parameterisation of the unit sphere.

$$\begin{aligned}
\int_{\partial B(a,r)} f(x)\, d\sigma(x) &= \sum_i \int_{U_i} (h_i f) \circ \Phi_i \,\mathrm{J}(\Phi_i)\, du \\
&= \sum_i \int_{U_i} (h_i f) \circ L \circ \tilde{\Phi}_i \,\mathrm{J}(L \circ \tilde{\Phi}_i)\, du \\
\mathrm{J}(L \circ \tilde{\Phi}_i)^2 &= \det {(L \circ \tilde{\Phi}_i)'}^T (L \circ \tilde{\Phi}_i)' = \det r^2 (\tilde{\Phi}_i')^T \tilde{\Phi}_i' \\
&= r^{2n-2} \det (\tilde{\Phi}_i')^T \tilde{\Phi}_i' = r^{2n-2} \mathrm{J}(\tilde{\Phi}_i)^2 \\
\int_{\partial B(a,r)} f(x)\, d\sigma(x) &= \sum_i \int_{U_i} (h_i (f \circ L)) \circ \tilde{\Phi}_i \, r^{n-1} \mathrm{J}(\tilde{\Phi}_i)\, du \\
&= r^{n-1} \int_{\partial B(0,1)} f \circ L(z)\, d\sigma(z) = r^{n-1} \int_{\partial B(0,1)} f(rz+a)\, d\sigma(z)
\end{aligned}$$

(b) Here we see the normalising constants:

$$M(c,a,r) = c \times \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(a,r)} d\sigma = c \times \frac{1}{n\omega_n r^{n-1}} \times n\omega_n r^{n-1} = c.$$

(c) These follow from the same properties of integrals. These are submanifold integrals, so maybe you want to write out the definition to see that the properties carry over from usual integrals.

(d) We can prove this with the $\varepsilon - \delta$ defintion of limits. Choose any $\varepsilon > 0$. By the defintion of continuity, there is a $\delta > 0$ so that for all $x \in B(a,\delta)$ we have $|f(x) - f(a)| < \varepsilon$. Then for all $0 < r < \delta$ we have

$$|M(f,a,r) - f(a)| = |M(f - f(a), a, r)| \le M(|f - f(a)|, a, r) \le M(\varepsilon, a, r) = \varepsilon.$$

This shows $\lim_{r\to 0^+} M(f,a,r) = f(a)$.

(e) We start with $M(v,x,r)$ and use part (a). We basically follow the proof of the mean value property, of which the main idea is to differentiate with respect to r to understand how the spherical mean changes with the radius

$$\begin{aligned}
\frac{\partial}{\partial r} M(v,x,r) &= \frac{1}{n\omega_n} \int_{\partial B(0,1)} \frac{\partial}{\partial r}(v(x+rz))\, d\sigma(z) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} \nabla v(x+rz) \cdot \frac{\partial}{\partial r}(x+rz)\, d\sigma(z) \\
&= \frac{1}{n\omega_n} \int_{\partial B(0,1)} \nabla v(x+rz) \cdot N\, d\sigma(z) = \frac{1}{n\omega_n} \int_{B(0,1)} \nabla \cdot \Big(\nabla v(x+rz)\Big)\, dz \\
&= \frac{r}{n\omega_n} \int_{B(0,1)} \triangle v(x+rz)\, dz.
\end{aligned}$$

Now we use the assumption that v is subharmonic, so the right hand side is positive. This means that $M(v,x,r)$ is an increasing function of r. Since we know that $\lim_{r\to 0} M(v,x,r) = v(x)$, it follows that $v(x) \le M(v,x,r)$.

(f) (I call this the local maximum principle:) Assume that a is a maximum of v. Then we have $v(a) \le M(v,a,r) \le M(v(a),a,r) = v(a)$, in other words $M(v(a) - v, a, r) = 0$. But we have seen in Exercise 5(e) (directly or as a consequence of the fundamental lemma of the calculus of variations) if a non-negative function like $v(a) - v$ has zero integral, then it is zero everywhere on the domain of integration. Hence $v(x) = v(a)$ for all $x \in B(a,r)$.

Now we can just use standard argument about connecting any two points of the domain, covering the path with overlapping balls, and repeatedly applying the local maximum principle to spread the fact that v is constant to the whole domain.

11. Fundamental solution of the Laplace equation.

Let $n \ge 2$. In this question we investigate a function known as the fundamental solution of the Laplace equation:

$$\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R},\ x \mapsto \begin{cases} -\frac{1}{2\pi} \log(\|x\|) & \text{for } n = 2 \\ \frac{1}{n\,(n-2)\,\omega_n} \|x\|^{2-n} & \text{for } n \ge 3 \end{cases}$$

(a) Let $u \in C^2(\mathbb{R}^n \setminus \{0\})$ be rotationally symmetric. This means that $u(x) = v(\|x\|)$ for some twice continuously differentiable function $v : [0, \infty) \to \mathbb{R}$. Show that such solutions of the Laplace equation $\triangle u = 0$ have the form $A\Phi + B$ for constants $A, B \in \mathbb{R}$.
Hint: Use the Laplacian in n-dimensions in spherical coordinates (look it up). *(2 Points)*

(b) Calculate

$$\nabla \Phi = -\frac{1}{n\,\omega_n} \frac{x}{\|x\|^n} \, .$$

(2 Bonus Points)

(c) The fundamental solution Φ is chosen from this set of solutions to have two properties. The first is that it vanishes at infinity ($B = 0$). The second property ($A = 1$) is that

$$\int_{\partial B(0,r)} \nabla \Phi \cdot N \, d\sigma = -1$$

for all radii. Verify this. *(2 Points)*

(d) Can you see an easy proof that the integral in the previous question is independent of the radius? *(2 Bonus Points)*

Why is this function called the fundamental solution? Because in terms of distributions $-\triangle \Phi = \delta$ (using Φ also as the name of the corresponding distribution F_Φ). We will now prove this.

(e) Check that Φ is locally integrable, so that it does indeed define a distribution. *(1 Point)*

(f) For a test function ψ, why does $-\triangle\Phi(\psi) = -\Phi(\triangle\psi)$? *(1 Point)*

(g) Separate the integral $-\Phi(\triangle\psi)$ into one part that contains the singularity from Φ and another part that is singularity free:

$$I_\varepsilon := -\int_{B(0,\varepsilon)} \Phi \triangle \psi,$$

$$J_\varepsilon := -\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi \triangle \psi.$$

Show that $\lim_{\varepsilon \to 0+} I_\varepsilon = 0$. *(2 Points)*

(h) For every J_ε, prove that the following estimate holds:

$$J_\varepsilon = -\int_{\partial B(0,\varepsilon)} \psi \, \nabla \Phi \cdot N \mathrm{d}\sigma + L_\varepsilon \, ,$$

where L_ε is some expression that converges to zero as $\varepsilon \to 0$.
[Hint: Green's second formula.] *(3 Points)*

(i) Finally, prove $\lim_{\varepsilon \to 0} J_\varepsilon = \psi(0)$. *(2 Points)*

In total we showed that $-\triangle\Phi(\psi) = \psi(0) = \delta(\psi)$ for all test functions ψ. This shows that the negative Laplacian of Φ as a distribution is indeed the delta distribution.

What is the fundamental solution good for? It gives us a solution of the Poisson equation $-\triangle u = f$ in $\mathbb{R}^n$.

(j) If F is a distribution with compact support, why does $U = F * \Phi$ solve the Poisson equation $-\triangle U = F$ in the sense of distributions? *(1 Point)*

(k) More generally, if L is a linear differential operator, a distribution Φ is called a fundamental solution if $L\Phi = \delta$. Give a solution to the inhomogeneous equation $LU = F$. *(1 Bonus Point)*

Solution.

(a) In case you didn't find a source, we rederive it here:

$$\begin{aligned}
\nabla(v(\|x\|)) &= v'(r)\nabla\|x\| = v'\frac{x}{\|x\|} \\
\triangle(v(\|x\|)) &= \nabla\cdot\left(v'\frac{x}{\|x\|}\right) = v''\frac{x}{\|x\|}\cdot\frac{x}{\|x\|} + v'\nabla\cdot\left(\frac{x}{\|x\|}\right) \\
&= v'' + v'\frac{\nabla\cdot x}{\|x\|} - v'\frac{x}{\|x\|^2}\cdot\nabla\|x\| \\
&= v'' + \frac{n-1}{r}v'.
\end{aligned}$$

Since u is a function of r alone, we see that we have an ODE to solve

$$\begin{aligned}
v'' + \frac{n-1}{r}v' &= 0 \\
r^{n-1}v'' + (n-1)r^{n-2}v' &= 0 \\
(r^{n-1}v')' &= 0 \\
r^{n-1}v' &= C \\
v &= B + C\int r^{1-n}\,dr.
\end{aligned}$$

(b) For both $n = 2$ (since $\omega_2 = \pi$) and $n \geq 3$ we have

$$\nabla\Phi = -\frac{1}{n\omega_n}\|x\|^{1-n}\nabla\|x\| = -\frac{1}{n\omega_n}x\|x\|^{-n}$$

(c)

$$\begin{aligned}
\int_{\partial B(0,r)} \nabla\Phi\cdot N\,d\sigma &= -\frac{1}{n\omega_n}\int_{\partial B(0,r)}\frac{x}{\|x\|^n}\cdot\frac{x}{\|x\|}\,d\sigma \\
&= -\frac{1}{n\omega_n}\int_{\partial B(0,r)}\frac{1}{r^{n-1}}\,d\sigma \\
&= -\frac{1}{n\omega_n r^{n-1}}\times n\omega_n r^{n-1} = -1.
\end{aligned}$$

(d) Except at $x = 0$ where it is not defined, Φ is harmonic. If Ω is any domain containing 0 with a submanifold boundary, there is a small ball $B(0,\varepsilon) \subset \Omega$. Then by the divergence theorem

$$0 = \int_{\Omega\setminus B(0,\varepsilon)}\triangle\Phi = \int_{\Omega\setminus B(0,\varepsilon)}\nabla\cdot\nabla\Phi = \int_{\partial\Omega}\nabla\Phi\cdot N\,d\sigma - \int_{\partial B(0,\varepsilon)}\nabla\Phi\cdot N\,d\sigma.$$

Given any two domains Ω and Ω' this shows that

$$\int_{\partial\Omega} \nabla\Phi \cdot N \, d\sigma = \int_{\partial\Omega'} \nabla\Phi \cdot N \, d\sigma.$$

The fact that it is constant of balls of different radii is just a special case.

(e) For $n \geq 3$ we have already checked this in Exercise 9(c). For the case $n = 2$ is comes down to the fact that $r \ln r$ is continuous at 0.

(f) By the definition of derivative of a distribution G

$$\triangle G(\psi) = \sum_i \partial_i^2 G(\psi) = \sum_i (-1)^2 G(\partial_i^2 \psi) = G\left(\sum_i \partial_i^2 \psi\right) = G(\triangle\psi).$$

(g) ψ is a test function, so $\triangle\psi$ is too. We can therefore bound it by its supremum. Since Φ is integrable, as the ball shrinks the integral vanishes.

(h) By Green's second formula and the fact that Φ is a harmonic function away from 0, we have

$$J_\varepsilon = -\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi\triangle\psi - \psi\triangle\Phi = \int_{\partial B(0,\varepsilon)} (\Phi\nabla\psi - \psi\nabla\Phi) \cdot N \, d\sigma,$$

where N is the outward pointing normal of the ball (it is the inward pointing normal of $\mathbb{R}^n \setminus B(0,\varepsilon)$ and therefore we have a sign change). The second term is the one we want, the first term we want to show vanishes. The integration is straightforward since Φ is constant on $\partial B(0,\varepsilon)$

$$\left| \int_{\partial B(0,\varepsilon)} \Phi\nabla\psi \cdot N \, d\sigma \right| \leq \|\nabla\psi\|_\infty \left|\Phi(\|x\| = \varepsilon)\right| n\omega_n \varepsilon^{n-1} \to 0.$$

(i) Continuing from the previous part and using part (c),

$$\begin{aligned} \left| -\int_{\partial B(0,\varepsilon)} \psi\nabla\Phi \cdot N \, d\sigma - \psi(0) \right| &= \left| -\int_{\partial B(0,\varepsilon)} \psi\nabla\Phi \cdot N \, d\sigma + \psi(0) \int_{\partial B(0,\varepsilon)} \nabla\Phi \cdot N \, d\sigma \right| \\ &\leq \int_{\partial B(0,\varepsilon)} |\psi(x) - \psi(0)| \, |\nabla\Phi \cdot N| \, d\sigma \\ &\leq \|\psi(x) - \psi(0)\|_{C^\infty(\partial B(0,\varepsilon))} \to 0. \end{aligned}$$

(j) From the derivative property over convolutions, we know that $-\triangle(F * \Phi) = F * (-\triangle\Phi) = F * \delta = F$.

(k) $U = F * \Phi$. The working is the same as the previous part, provided everything is well defined.